# BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review of Mathematical Science

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## Whole No. 241

## CONTENTS


PAGE

The Twenty-Second Summer Meeting of the American Mathematical Society. By Dr. THOMAS BUCK - - - 1

Groupless Triad Systems on Fifteen Elements. By Dr. LOUISE D. CUMMINGS and Professor H. S. WHITE - - 12

Note on Green's Theorem. By Mr. C. A. EPPERSON - - 17

Convergence of the Series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^i y^j}{i-j\gamma}$ ($\gamma$ Irrational). By Professor W. D. MACMILLAN - - - - - 26

A Certain Class of Functions Connected with Fuchsian Groups. By Professor ARNOLD EMCH - - - - 33

Professor Bôcher's Views Concerning the Geometry of Inversion. By Professor EDUARD STUDY. With Note by Professor MAXIME BÔCHER - - - - - - 38

The Davis Calculus. By Professor E. W. DAVIS - - - 41

Notes - - - - - - - - - - - - 41

New Publications - - - - - - - - 48


## Whole No. 242


PAGE

On the Relation between Linear Algebras and Continuous Groups. By Professor L. E DICKSON - - - - 53

An Aspect of the Linear Congruence with Applications to the Theory of Fermat's Quotient. By Mr H. S. VANDIVER - - - - - - - - - - 61

Limits of the Degree of Transitivity of Substitution Groups. By Professor G. A. MILLER - - - - - - - 68

The Permutations of the Natural Numbers Can not be Well Ordered. By Professor A. B. FRIZELL - - - 71

Relations among Parameters along the Rational Cubic Curve. By Professor J. E. ROWE - - - - - - - 74

Vallée Poussin's Cours D'Analyse. By Professor M. B. PORTER - - - - - - - - - - 77

Enumerative Geometry. By Dr. E. S. ALLEN - - - 85

Mathematics in Australia. By Professor R C. ARCHIBALD 94

A Correction. By Professor R. C. ARCHIBALD - - - 100

Shorter Notice - - - - - - - - - - 100

Notes - - - - - - - - - - - - 101

New Publications - - - - - - - - - 104

## Whole No. 243

## CONTENTS


PAGE

Concerning Absolutely Continuous Functions. By Professor M. B. PORTER - - - - - - - - - - 109

On the Representation of Numbers in the Form $x^3+y^3+z^3-3xyz$. By Professor R. D. CARMICHAEL - 111

On the Linear Continuum. By Dr. R. L. MOORE - - 117

A Problem in the Kinematics of a Rigid Body. By Professor PETER FIELD - - - - - - - - 122

Jules Henri Poincaré. By Professor R. C. ARCHIBALD - 125

Shorter Notices - - - - - - - - - - - 136

Notes - - - - - - - - - - - - 151

New Publications - - - - - - - - - 155


## Whole No. 244

## CONTENTS


PAGE

The October Meeting of the American Mathematical Society. By Professor F. N. COLE - - - - - - - - 161

The Twenty-Seventh Regular Meeting of the San Francisco Section. By Dr. THOMAS BUCK - - - - - - 172

Transformation Theorems in the Theory of the Linear Vector Function. By Dr. V. C. POOR - - - - 174

Napier's Descriptio and Constructio. By Professor R. C. ARCHIBALD - - - - - - - - - - - 182

Mathematical Quotation Books. By Professor R. C. ARCHIBALD - - - - - - - - - - - - 188

Shorter Notices - - - - - - - - - - - 192

Notes - - - - - - - - - - - - 204

New Publications - - - - - - - - - 209

## Whole No. 245

## CONTENTS


PAGE

The Ninth Regular Meeting of the Southwestern Section. By Professor O. D. KELLOGG - - - - - - 215

A Note on the Problem of Lagrange in the Calculus of Variations. By Professor G. A. BLISS - - - - - 220

Concerning a Non-Metrical Pseudo-Archimedean Axiom By Dr. R. L. MOORE - - - - - - - - - 225

A Type of Singular Points for Transformation of Three Variables. By Dr. W. V. LOVITT - - - - - 236

The History of the Construction of the Regular Polygon of Seventeen Sides. By Professor R. C. ARCHIBALD - 239

Shorter Notices - - - - - - - - - - - 246

Notes - - - - - - - - - - - - 255

New Publications - - - - - - - - - - 258


## Whole No. 246

## CONTENTS


PAGE

The Twenty-Second Annual Meeting of the American Mathematical Society. By Professor F. N. COLE - - - 263

Winter Meeting of the Society at Columbus. By Professor H. E SLAUGHT - - - - - - - - - 280

On Pierpont's Definition of Integrals. By Professor M. FRÉCHET - - - - - - - - - - - 295

Reply to Professor Fréchet's Article. By Professor JAMES PIERPONT - - - - - - - - - - 298

Carmichael's Monographs on Branches of the Theory of Numbers. By Professor L. E. DICKSON - - - 303

Notes - - - - - - - - - - - - 311

New Publications - - - - - - - - - - 313

## Whole No. 247

## CONTENTS


PAGE

Some Remarks on the Historical Development and the Future Prospects of the Differential Geometry of Plane Curves. By Professor E. J. WILCZYNSKI - - - 317

A Certain System of Linear Partial Differential Equations. By Dr. H. BATEMAN - - - - - - - - - 329

Changing Surface to Volume Integrals. By Professor E. B. WILSON - - - - - - - - - - - 336

A New Method of Finding the Equation of a Rational Plane Curve from its Parametric Equations. Professor J. E. ROWE - - - - - - - - - - - 338

The Physicist J. B. Porta as a Geometer. By Professor GINO LORIA - - - - - - - - - - - 340

Pierpont's Functions of a Complex Variable. By Professor H. P. MANNING - - - - - - - - - 343

Shorter Notices - - - - - - - - - - 350

Notes - - - - - - - - - - - 365

New Publications - - - - - - - - - 369


## Whole No. 248

## CONTENTS


PAGE

The February Meeting of the American Mathematical Society. By Professor F. N. COLE - - - - - - 373

On a Configuration on Certain Surfaces. By Professor C. H. SISAM - - - - - - - - - - 381

On Separated Sets. By Professor W. A. WILSON - - 384

Singular Points of Transformations and Two-Parameter Families of Curves. By Dr. W. V. LOVITT - - - 387

An Elementary Boundary Value Problem. By Professor DUNHAM JACKSON - - - - - - - - - 393

Concerning Reviews. By Professor R. E. Moritz, with Reply by Professor R. C. ARCHIBALD - - - - 398

Shorter Notices - - - - - - - - - - 402

Notes - - - - - - - - - - - 415

New Publications - - - - - - - - - 420

## Whole No. 249

## CONTENTS


PAGE

The April Meeting of the Society at Chicago. By Professor Arnold Dresden - - - - - - - - 425

Note on Functions of Several Complex Variables. By Professor W. F. Osgood - - - - - - - - 443

Quasi-Periodicity of Asymptotic Plane Nets. By Dr. A. L. Nelson - - - - - - - - - - 445

Concerning Hill's Derivation of the Lagrange Equations of Motion. By Professor K. P. Williams - - - 455

A Simplification of the Whitehead-Huntington Set of Postulates for Boolean Algebras. By Dr. B. A. Bernstein 458

Note on Regular Transformations. By Dr. L. L. Silverman 459

Shorter Notices - - - - - - - - - - 461

Notes - - - - - - - - - - - 472

New Publications - - - - - - - - - 476


## Whole No. 250

## CONTENTS


PAGE

The April Meeting of the American Mathematical Society in New York. By Professor F. N. Cole - - - 481

Application of an Equation in Variable Differences to Integral Equations. By Professor G. C. Evans - - 493

Operators in Vector Analysis. By Dr. V. C. Poor - - 503

Shorter Notices - - - - - - - - - - 504

Notes - - - - - - - - - - - 514

New Publications - - - - - - - - - 516

Twenty-Fifth Annual List of Published Papers - - - 520

Index of Volume XXII - - - - - - - - 529

BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

## THE TWENTY-SECOND SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-second summer meeting of the Society was held at the University of California on Tuesday, and at Stanford University on Wednesday, August 3–4, 1915, in connection with the Panama-Pacific International Exposition. The following thirty-four members of the Society were in attendance upon the two sessions:

Professor R. E. Allardice, Dr. Nathan Altshiller, Dr. Charlotte C. Barnum, Dr. A. A. Bennett, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Professor C. E. Brooks, Dr. Thomas Buck, Professor E. F. A. Carey, Professor E. E. De Cou, Dr. H. B. Curtis, Professor G. C. Edwards, Dr. Elizabeth B. Grennan, Professor F. L. Griffin, Professor F. W. Hanawalt, Professor Charles Haseman, Professor M. W. Haskell, Professor E. R. Hedrick, Professor L. M. Hoskins, Dr. Frank Irwin, Professor C. J. Keyser, Professor D. N. Lehmer, Professor J. H. McDonald, Professor W. A. Manning, Professor H. C. Moreno, Professor R. E. Moritz, Professor L. I. Neikirk, Professor C. A. Noble, Professor E. W. Ponzer, Mr. F. D. Posey, Professor T. M. Putnam, Professor H. W. Stager, Professor H. W. Tyler, Professor S. E. Urner.

Professor M. W. Haskell, chairman of the San Francisco Section, presided at the session on Tuesday afternoon, and Professor R. E. Allardice at that on Wednesday afternoon.

Tuesday morning was devoted to a joint session with the American Astronomical Society and Section A of the American Association for the Advancement of Science. Addresses were delivered by Professors C. J. Keyser on "The human significance of mathematics," and G. E. Hale on "The work of a modern observatory." The astronomers, mathematicians, and physicists lunched at the Faculty Club as guests of Professors Leuschner, Haskell, and E. P. Lewis.

The social programme included a dinner with the American Astronomical Society at the Hotel Oakland on Wednesday evening, an excursion to the Lick Observatory on Friday, and a luncheon given by Mrs. Phoebe Hearst at the Hacienda del Pozo de Verona on Saturday.

The following papers were presented at this meeting:

(1) Professor L. E. DICKSON: "Invariantive classification of pairs of conics modulo 2."

(2) Professor C. J. DE LA VALLÉE POUSSIN: "Sur l'intégrale de Lebesgue."

(3) Mr. A. R. SCHWEITZER: "On the solution of a class of functional equations."

(4) Dr. NATHAN ALTSHILLER: "On the circles of Apollonius."

(5) Dr. DUNHAM JACKSON: "Proof of a theorem of Haskins."

(6) Dr. W. W. KÜSTERMANN: "Fourier constants of functions of two variables."

(7) Dr. B. A. BERNSTEIN: "A set of four independent postulates for Boolean algebras."

(8) Professor G. A. MILLER: "Limits of the degree of transitivity of substitution groups."

(9) Professor R. D. CARMICHAEL: "On the representation of numbers in the form $x^3 + y^3 + z^3 - 3xyz$."

(10) Professor H. S. WHITE: "Seven points on a gauche cubic curve."

(11) Professor M. W. HASKELL: "The del Pezzo quintic curve."

(12) Professor L. J. RICHARDSON: "A phase of Roman mathematics."

(13) Dr. C. A. FISCHER: "Functions of surfaces with exceptional points or curves."

(14) Mr. A. R. WILLIAMS: "On a birational transformation connected with a pencil of cubics."

(15) Professor F. N. COLE: "Note on the triad systems in 15 letters."

(16) Professor A. B. COBLE: "The determination of the lines on a cubic surface."

(17) Mr. H. S. VANDIVER: "An aspect of the linear congruence, with applications to the theory of Fermat's quotient."

(18) Dr. C. H. FORSYTH: "An interpolation formula based upon central and multiple differences."

(19) Dr. G. M. GREEN: "On isothermally conjugate nets of space curves."

(20) Professor L. P. EISENHART: "Surfaces of rolling and transformations of Ribaucour."

(21) Mr. A. R. SCHWEITZER: "Generalized quasi-transitive functional relations."

(22) Professor L. M. HOSKINS: "'Quantity of matter' in dynamics."

(23) Dr. A. A. BENNETT: "The iteration of functions of one variable."

Professor Richardson was introduced by Professor Haskell and Mr. Williams by Professor Lehmer. Professor de la Vallée Poussin's paper was communicated to the Society through Professor Birkhoff. Professor White's paper was read by Professor Haskell. The papers by Professor Dickson, Professor de la Vallée Poussin, Mr. Schweitzer, Dr. Jackson, Dr. Küstermann, Professor Miller, Professor Carmichael, Dr. Fischer, Professor Cole, Professor Coble, Mr. Vandiver, Dr. Forsyth, Dr. Green, and Professor Eisenhart were read by title.

Abstracts of the papers, including Professor Keyser's address, follow below. The abstracts are numbered to correspond to the papers in the list above.

In Professor Keyser's address the treatment, conducted in the spirit appropriate to an international exposition, aims at being interesting and intelligible to the general educated public. A sketch of the manner in which a historian of mathematics might hope to vindicate the science is followed by a more elaborate exposition of the task of attaining the same end through an account of the present state of the science. The claims of mathematics to human regard as based on its applications in a wide range of other sciences and arts are next dealt with. This matter is followed by a characterization of the modern critical movement of the science and of the resulting conception of the distinctive character of mathematics, especially in its relation to modern developments in logic. The chief emphasis of the address falls on the bearings of the science as distinguished from its applications—upon the significance of mathematics for man regarded as finding his supreme interest in seeking permanent values in an inpermanent world.

1. With a conic $F$ modulo 2 is associated covariantively a point $A$, called its apex, and a unique line $L$, and conversely $A$ and $L$ uniquely determine $F$ (Madison Colloquium Lectures, 1914, page 69). Hence the projective classification of pairs of conics $F$ and $F'$ is equivalent to that of the systems $A, L, A', L'$ of two points and two lines and the degenerate systems in which one or more of the four elements are absent. A simple geometrical discussion of such systems leads Professor Dickson to the theorem: Two pairs of conics modulo 2 are projectively equivalent if and only if they have the same properties as regards existence of apices and covariant lines, distinctness of apices and lines, and incidence of apices and lines. These properties are expressed analytically by very simple modular invariants, which therefore form a fundamental system of modular invariants of two conics.

2. Professor de la Vallée Poussin's paper, which reproduces a part of his recent course of lectures at Harvard University, will appear in the *Transactions*.

3. The object of Mr. Schweitzer's paper is to discuss the solution of the equations

$$f\{\lambda_1 f(t_1, t_2, \cdots, t_n, x_1), \lambda_2 f(t_1, t_2, \cdots, t_n, x_2), \cdots,$$
$$\lambda_{n+1} f(t_1, t_2, \cdots, t_n, x_{n+1})\} = \mu f\{x_{i_1}, x_{i_2}, \cdots, x_{i_{n+1}}\},$$

where $n = 1, 2, 3$, etc.; the subscripts $i_1, i_2, \cdots, i_{n+1}$ are distinct and range respectively over the values $1, 2, \cdots, n+1$; $\lambda_i$ and $\mu$ denote functions of a single variable. The $(n+1)!$ functional equations thus defined may be put into $(1, 1)$ relation with the substitutions on $n+1$ symbols and therefore conveniently denoted by the notation

$$E\left\{\begin{pmatrix} 1 & 2 & 3 & \cdots & (n+1) \\ i_1 & i_2 & i_3 & \cdots & i_{n+1} \end{pmatrix}; \lambda_1(x), \lambda_2(x), \cdots, \lambda_{n+1}(x), \mu(x)\right\}.$$

For $n = 1$ the equations have been completely discussed by the author in previous articles. For $n > 1$ the special instance of identity of some or all of the functions $\lambda_i(x)$, $\mu(x)$ is considered. In the simplest case, namely, when $n = 2$ and the functions $\lambda_i(x)$, $\mu(x)$ are all identical and equal to $x$, the following theorems on the equations

$$E\left\{\begin{pmatrix} 1 & 2 & 3 & \cdots & (n+1) \\ i_1 & i_2 & i_3 & \cdots & i_{n+1} \end{pmatrix}\right\}$$

are valid:

1. $E\{(1)\ (2)\ (3)\}$ implies the existence of $\psi(x)$ such that $f(x_1, x_2, x_3) = \psi^{-1}\{[\psi(x_3) - \psi(x_2)] + \theta[\psi(x_2) - \psi(x_1)]\}$, where $\theta(x)$ is arbitrary.
2. $E\{(1)\ (23)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{-2\psi(x_1) + \psi(x_2) + \psi(x_3)\}$.
3. $E\{(2)\ (13)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{\psi(x_1) - 2\psi(x_2) + \psi(x_3)\}$.
4. $E\{(3)\ (12)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{[\psi(x_3) - \psi(x_2)] + \theta[\psi(x_2) - \psi(x_1)]\}$, $\theta(x) = x + \theta(-x)$.
5. $E\{(123)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{\psi(x_1) + c\psi(x_2) + c^2\psi(x_3)\}$, where $1 + c + c^2 = 0$.
6. $E\{(132)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{c\psi(x_1) + \psi(x_2) + c^2\psi(x_3)\}$, where $1 + c + c^2 = 0$.

Solutions 2–4 are special cases of the Solution 1.

4. In his contribution to the modern geometry of the triangle Dr. Altshiller, starting with the usual definition of the circles of Apollonius connected with a triangle, derives, in a purely synthetic way, a series of properties, partly known, involving the Lemoine point, the symmedian lines, the Brocard diameter, etc., and leads up to the theorem: The center of any one Apollonian circle is a center of similitude of the two other circles of Apollonius, the second center of similitude being the pole of the Brocard diameter with respect to the first circle. This paper will appear in the *American Mathematical Monthly*.

5. It is a theorem due to Haskins that if $f$ and $\varphi$ are two bounded functions such that the definite integral of $f^n$ over an interval is equal to that of $\varphi^n$ for all positive integral values of $n$, the set of points where $\alpha \leqq f \leqq \beta$ and the set where $\alpha \leqq \varphi \leqq \beta$ have the same measure, for any pair of numbers $\alpha, \beta$. Dr. Jackson gives a short proof of this theorem, based on a polynomial approximation and Lebesgue's theorem on the integration of a uniformly bounded sequence. The result is of particular interest in the case where the functions are monotone.

6. In volume 57 of the *Mathematische Annalen* A. Hurwitz has shown how to express the product of two ordinary Fourier

series in form of another Fourier series, or, stated differently, how to compute the Fourier constants of the product of two functions from those of the functions themselves. Dr. Küstermann solves the analogous problem for double Fourier series. The solution depends vitally upon the proof of the relation

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[f(x,y)]^2dxdy = \sum_{\mu,\nu=0}^{\infty}\frac{(a^2+b^2+c^2+d^2)_{\mu\nu}}{2^{E[1/(\mu+1)]+E[1/(\nu+1)]}},$$

where $(a, b, c, d)_{\mu\nu}$ are the Fourier constants of $f(x, y)$ presumed to be bounded and integrable in the Riemann sense. The analogue of this relation for functions of a single variable is due to Parseval and was proved by him under certain restrictions on the nature of convergence of the Fourier series involved. In 1893 de la Vallée Poussin gave a proof requiring merely that the function and its square be integrable. Hurwitz, in 1903, again called attention to the wide scope of the theorem, gave a new proof, and applied it to the problem mentioned. More recently the formula has gained in general interest through the researches of Riesz and Fischer (Riesz-Fischer theorem). In the present paper the proof is established by means of Poisson's double integral.

7. The most economical set of postulates for the logic of classes is that presented by H. M. Sheffer in the *Transactions* of October, 1913. Using as basis the operation of "rejection," which Dr. Sheffer used, Dr. Bernstein obtains a set of four postulates from which the former's set of five can be deduced. The paper will appear in the *Transactions*.

8. The main theorem established by Professor Miller in the present paper is as follows: A substitution group of degree $n = kp + r$, where $p$ is a prime number and $r$ and $k$ are positive integers such that $p > k$ and $r > k$, cannot be more than $r$ times transitive unless either this group includes the alternating group of degree $n$ or $k = 1$ and $r = 2$. By means of this theorem it can easily be proved that a group which does not include the alternating group of its degree cannot be as much as 15-fold transitive unless its degree exceeds 1,000. It is also proved that whenever $n > 12$ the theorem will always give a smaller upper limit for the degree of transitivity of a substitution group which does not include the

alternating group of its degree than the limit obtained by using the well-known formula $\frac{1}{2}n + 1$. The present paper is an extension of results obtained by the same writer in an article published in the BULLETIN, volume 4 (1898), page 140, and it aims to replace the formula $\frac{1}{2}n + 1$ for the upper limit of the degree of transitivity of the groups in question by a theorem which gives at least as low limits when $n > 7$, and gives always a lower limit when $n > 12$.

9. Among the theorems presented by Professor Carmichael in this note are the following: Every prime number other than 3 can be represented in one way and in only one way in the form $x^3 + y^3 + z^3 - 3xyz$, where $x, y, z$ are non-negative integers; primes of the form $6n + 1$ have one additional representation in which at least one of the integers $x, y, z$ is negative.

10. In previous theories seven points on a twisted cubic curve have been studied only in combinations that were symmetric; and the only theorem known, beyond what may be considered as definition of the curve, states that the 35 points where the seven osculating planes meet must lie on a surface of order five. Professor White examines sets of seven derived points corresponding to a triad system on seven elements, and finds them to lie upon a second cubic curve, one of thirty determined by the first seven points. Analytic proofs are given.

11. Professor Haskell's paper contains the following items: Reduction of the Hessian of the plane quintic with five cusps, showing that the five cusps and five inflexions lie on a cubic, and that the inflexions are rational in terms of the cusps. Determination of a conic with regard to which the quintic is self-dual, of a Cremona transformation of the quintic into itself, and of sets of conics and cubics connected with the cusps and inflexions.

12. Professor Richardson's paper gives a practical illustration of manual multiplication as practised by the Romans, together with a brief historical survey of this system of reckoning.

13. In a paper published in the July, 1914, number of the *American Journal of Mathematics*, Dr. Fischer has given a

definition of the derivative of a function of a surface analogous to Volterra's definition of the derivative of a function of a line, and has proved that if the derivative is continuous and approached uniformly, the first variation of the function is equal to the double integral of the derivative multiplied by the first variation of the dependent variable. In the present paper this theory is extended to the case where there are points or curves where the derivative does not exist. The theory of exceptional points is used in finding the second variation of a function of a double integral, and the theory of exceptional curves is applied to the variable boundary problem in the calculus of variations.

14. The theorem that all the cubics through eight points pass through a ninth, suggests the following transformation which is evidently birational: Fix seven of the eight points, and make correspond to a variable eighth the ninth point through which all the cubics pass.

Dr. Hart (see *Cambridge and Dublin Mathematical Journal* volume 6, page 181) has given a synthetic construction for the ninth point determined by eight given points. By an analytic treatment of this construction Mr. Williams has found the following relation between the coordinates, $x'$, $y'$, $z'$, of 9 and the coordinates of 8: $x' = C_1C_2K_1$; $y' = C_2C_3K_2$; $z' = C_3C_1K_3$, where the $C$'s are cubic functions and the $K$'s are quadratic. These equations define a Cremona transformation.

In the above transformation there are infinitely many invariant points, all of which lie on a sextic curve of deficiency three, having double points at each of the seven fixed points. At each of these double points the tangents to the sextic coincide with the tangents to the nodal cubic having the same double point and passing through the other six fixed points. The sextic also meets the line determined by any two of these seven points in the points where this line meets the conic determined by the other five. From this follows the remarkable theorem concerning seven arbitrary points in the plane: if the line joining any two of them be cut by the conic determined by the other five, the forty-two points thus obtained lie on a sextic. This sextic satisfies seventy-seven conditions, almost three times the number necessary to determine it.

By introducing relations among the seven given points there result other types of Cremona transformations with their corresponding invariant curves. Thus if three of the seven points are on a line, we have a transformation of the seventh degree with an invariant quintic curve. And this quintic satisfies fifty-three conditions. Similarly, if six of the seven points lie on a conic, there results a quartic transformation with an invariant quartic curve fulfilling forty-seven conditions. If six points are on a conic and at the same time three are on a line, a cubic transformation results with an invariant cubic curve fulfilling twenty-nine conditions. If six points are on a conic and if at the same time three are on one straight line and three on another, the transformation is quadratic, and the invariant curve is a conic which fulfills sixteen conditions. This quadratic transformation is of a type of which inversion with respect to the unit circle is a special case.

15. Professor Cole has found, by detailed examination of the various cases, that every triad system in 15 letters, with a single exception, presents an interlacing, $(xab)(xcd)$, $(yac)(ybd)$, the exceptional case being one of Heffter's systems. It is also found that every triad system in 15 letters presents either a hexad, $(xab)(xcd)(xef)$, $(yac)(yde)(ybf)$, or triple tetrads, $(xab)(xcd)(xef)(xgh)(xij)(xkl)$, $(yac)(ybd)(yeg)(yfh)(yik)(yjl)$.

16. In earlier reports Professor Coble has discussed the Cremona group $G_{6,2}$ in $\Sigma_4$ of order 51840 determined by a $P_6^2$ (six points in a plane). It is the purpose of this paper to show that $G_{6,2}$ furnishes the direct algebraic connection between the cubic surface $C^3$, mapped on the plane by cubic curves on $P_6^2$, and the collineation groups which arise in the trisection of the theta functions of genus 2. This connection is made by the use of irrational invariants of $C^3$ which are associated with the separation of the lines of the surface. The discriminant of $C^3$ consists of 36 irrational factors corresponding to the 36 double sixes of $C^3$. By forming products of 9 such factors properly selected, a set of 40 irrational "complex invariants" is obtained. These determine in $\Sigma_4$ a set of 40 quintic spreads conjugate under $G_{6,2}$. The spreads however lie in a linear system of dimension 9 and one can

choose a linearly independent set, $X_0, \cdots, X_4$; $U_0, \cdots, U_4$, with the striking property developed below.

On the other hand Klein had discovered and Burkhardt had developed a collineation group in $S_4$ with variables $Y_0, \cdots, Y_4$ of order 25920. If to this group there be added the correlations which transform it into itself there is obtained a mixed group $g_{51840}$ in the contragredient variables $Y_0, \cdots, Y_4$; $V_0, \cdots, V_4$, which is isomorphic with $G_{6,2}$ and which has the following property: The variables $Y$, $V$ are transformed under $g_{51840}$ precisely as the irrational invariants $X$, $U$ above are transformed under $G_{6,2}$.

This property leads to the rational determination of the lines of a cubic surface (after the adjunction of the square root of its discriminant) in terms of the solution of the form problem of Burkhardt's group without passing beyond the domain of the irrational invariants of the given surface. The methods hitherto suggested for this purpose required the introduction of an equation of degree 27 (or a resolvent of some other degree), i. e., the introduction of a binary field necessarily extraneous to the surface itself.

17. In 1903, Professor G. D. Birkhoff communicated to Mr. Vandiver the following theorem:

If $p$ is a prime integer and $a$ is a positive integer prime to $p$, then there is at least one and not more than two sets $(x, y)$ such that

$$a \equiv \pm x/y \pmod{p},$$

where $x$ and $y$ are integers prime to each other and $0 < x < \sqrt{p}$, $0 < y < \sqrt{p}$.

In the present paper a proof of this theorem is given, involving a continued fraction algorithm for the direct determination of each set. It is also shown that the fact that there exists at least one set $(x, y)$ follows from a theorem due to Minkowski. The least positive residues of the integers

$$1^{p-1}, \quad 2^{p-1}, \cdots, \quad (p-1)^{p-1}$$

modulo $p^2$ are termed *proper residues* modulo $p^2$. By an extension of the result mentioned above the following theorem is derived:

There are not more than

$$p - \tfrac{1}{2}(1 + \sqrt{2p-5})$$

and not less than $[\sqrt{p}]$ incongruent proper residues modulo $p^2$, if $p$ is a prime $> 2$.

Applications to problems in connection with Fermat's last theorem are then discussed.

18. The accuracy of an interpolation based upon finite differences is increased as the values of the independent variable occurring in the differences approach the value of the independent variable of the ordinate which is interpolated. Dr. Forsyth has derived an interpolation formula wherein these values are identical.

Furthermore, instead of the usual averaging of differences (i. e., use of first differences) to supply differences which are missing (as always happens in central differences), differences are used whose orders are in keeping with the order of the interpolation formula itself.

19. In Dr. Green's note is given a new geometric interpretation of Bianchi's condition for isothermally conjugate nets of curves on a surface. It will appear in the *Proceedings of the National Academy of Sciences.*

20. In a note published in volume 23 of the *Rendiconti dei Lincei* Bianchi has defined as a surface of rolling the surface described by a point invariably fixed with respect to a surface $S_0$ as the latter rolls over an applicable surface $S$, which Bianchi calls the surface of support. He shows that, given any surface $\Sigma$, the problem of finding pairs of applicable surfaces $S_0$ and $S$ such that $\Sigma$ is a surface of rolling as $S_0$ rolls over $S$ reduces to the integration of a partial differential equation of the second order and of a Riccati equation. Two surfaces are said to be in the relation of a transformation of Ribaucour when they constitute the envelope of a two-parameter family of spheres such that the lines of curvature correspond on the two surfaces, corresponding points being on the same sphere. Professor Eisenhart has shown that the necessary and sufficient condition that either surface of two so related be a surface of rolling with the locus of the centers of the spheres for surface of support is that the two surfaces be isothermic and in the relation of a so-called transformation $D_m$, discovered by Darboux.

21. As a quasi-transitive functional relation in the most general sense, Mr. Schweitzer defines

$$\phi\{f_1(t_1, t_2, \cdots, t_n, x_1), f_2(t_1, t_2, \cdots, t_n, x_2), \cdots, f_{n+1}(t_1, t_2, \cdots, t_n, x_{n+1})\} = \psi(x_1, x_2, \cdots, x_{n+1}),$$

where $n = 1, 2, 3$, etc. From this class of equations other classes are deduced, first, by the homologous transposition of the $x$'s in the left hand member and, second, by substituting $x$'s for some or all of the $t$'s in the set of equations thus obtained (including the original class). For example, one obtains by homologous transposition of the $x$'s when $n = 2$

$$\phi\{f_1(x_1, t_1, t_2), f_2(x_2, t_1, t_2), f_3(x_3, t_1, t_2)\} = \psi(x_1, x_2, x_3)$$
$$\phi\{f_1(t_1, x_1, t_2), f_2(t_1, x_2, t_2), f_3(t_1, x_3, t_2)\} = \psi(x_1, x_2, x_3).$$

Interesting problems are presented by systems of equations belonging to the same or different classes.

22. The proposition that the accelerations of different bodies under the action of equal forces are inversely proportional to their masses is often asserted to be merely a definition of mass. The object of the paper by Professor Hoskins is to show that, in applying this proposition, our interpretation of it involves the notion of mass as a quantitative measure of the matter of which the bodies are composed.

23. Dr. Bennett considers several topics connected with the iteration of functions of one variable. Matrices with an infinite number of elements are used to obtain a classification of the types of series which are to be distinguished with respect to the nature of their iteration. The different types are considered, with particular reference to questions of convergence. The iteration of functions of a real variable is also considered. The paper will appear in the *Annals of Mathematics*.

THOMAS BUCK,
*Acting Secretary.*

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## GROUPLESS TRIAD SYSTEMS ON FIFTEEN ELEMENTS.

BY DR. LOUISE D. CUMMINGS AND PROFESSOR H. S. WHITE.

(Read before the American Mathematical Society April 24, 1915.)

FROM previous publications and a paper presented to this Society in October, 1914, 44 different triad systems on 15 elements ($\Delta_{15}$) are known. These 44 systems separate into two types 23: of the systems each contain one or more systems

on 7 elements and have been designated as headed systems, the remaining 21 contain no $\Delta_7$ and have been designated as headless systems. The groups for the 44 systems have been obtained and range in order from 8!/2 down to 2. Hence for each of these systems there exists at least one substitution different from identity which transforms the system into itself.

Systems whose group is identity or *groupless* systems on 15 elements have not been known hitherto. No groupless systems exist for 7, 9, or 13 elements. In a paper published in the *Transactions*, January, 1915, one of us has proved the existence of millions of triad systems on 31 elements with the group identity. This raised the question concerning the existence of the groupless systems for 15 elements. About six months ago we began a rigorous search for this type of $\Delta_{15}$, and, while the investigation is at present far from completion, we have had the good fortune to discover systems of this new species.

In the *Transactions* for January, 1913, Professor Cole discussed triad systems on 13 elements, and pointed out the fact that two elements in a system may be connected by what he has termed an "interlacing"; for example the four triads $a12$, $a34$, $b13$, $b24$ occurring in a system form an interlacing of the elements $a$ and $b$. Interlacing and some extensions of this idea have proved to be of fundamental importance in the formation of the groupless systems.

All interlacings and the more extended forms of interconnection of the elements, which we make use of, may be obtained by applying to the system under consideration what may be called a contracted form of sequences and indices. The $\Delta_{15}$ is written in a 15 by 7 array. Each element heads one column; below it are placed the 7 pairs of elements which with the element at the head occur in the triads of the system. Heretofore, sequences and indices have been formed from the 3 columns of a triad in any $\Delta_{15}$. The same process is now applied to every pair of columns of a $\Delta_{15}$. Since the number of combinations of 15 columns, two at a time, is 105, 105 pairs of columns must be examined, unless the group for the system is already known and is different from identity. If the group contains an operator of order $k$, then $k$ pairs of columns are examined simultaneously. The process may be illustrated in its application to a system $J$, with a group of order 4 generated by $s = (a)(bc)(d8e7)\ (f3g4)(1526)$. Pairs of

columns selected from the following table show every type of contracted index that occurs in any system.

| *a* | *b* | *d* | 1 |
|---|---|---|---|
| *de* | *df* | *ae* | *a*2 |
| *fg* | *eg* | *bf* | *b*3 |
| 12 | *ac* | *c*2 | *ce* |
| 34 | 13 | *g*8 | *d*5 |
| 56 | 24 | 15 | *f*8 |
| 78 | 57 | 37 | *g*4 |
| *bc* | 68 | 46 | 67 |

| Pair of Columns. | Index. | Contracted Sequences. |
|---|---|---|
| *ab* | $2^3$ | *de/gf/d*, 12/43/1, 56/87/5; |
| *b*1 | $3^2$ | *df*/86/75/*d*, 24/*ge*/*ca*/2; |
| *da* | 6 | *c*2/15/64/37/8*g*/*fb*/*c*; |
| *db* | 2, 4 | *ae*/*g*8/64/2*c*/*a*, 15/73/1. |

The substitution $s$ applied to the pair of columns *ab* gives the pair *ac* with the same index and similar sequences. If $s$ is applied to the pair of columns *b*1, the three pairs *c*5, *b*2, *c*6 are obtained with the index $3^2$. The analysis of the 105 pairs of columns shows that the contracted indices $2^3$; 2, 4; $3^2$; 6 belong, respectively, to 2, 24, 4, and 75 pairs of the columns.

The new groupless systems are formed by interchanging duads of one column with those of another column. For example, in the pair of columns *ab*, the duads *de*, *fg* of column *a* may be interchanged with *df*, *eg* of column *b*; such an interchange involving four elements, contained only in two pairs in each column, shall be designated as a *quadrangular transformation*. The columns *d*, *e*, *f*, *g* must now be re-written, in agreement with the new triads introduced into the system, and the undisturbed 9 columns of *J* with the re-constructed 6 columns form a new system *JQ*. The four duads 12, 34, 56, 78 of column *a* might be interchanged with the four duads on the same elements in column *b*, forming an octagonal transformation, but this is equivalent to the above quadrangular transformation followed by the interchange of the elements *a* and *b*.

In the pair of columns *b*1 the duads *df*, 68, 57 of column *b* may be interchanged with the duads *f*8, 67, *d*5 of column 1; such an interchange, involving six elements appearing exclusively in 3 pairs in each of the two columns, shall be designated as a *hexagonal transformation*. New columns *d*, *f*, 6, 8, 5, 7

must next be constructed; these eight reconstructed columns, with the seven undisturbed columns of $J$, form a system $JH$. The application of the second hexagonal transformation in $b1$ is equivalent to an application of the first hexagonal transformation followed by the interchange of the elements $b$ and 1.

A transformation on 12 elements simply interchanges the two elements which head the columns.

Therefore only the quadrangular and the hexagonal transformations which exist in a system require consideration.

By means of the operators of the group of the system, the minimum number of non-congruent transformations of each of the above types is determined—for example, in $J$, the 8 hexagonal transformations reduce to 1, and the 30 quadrangular transformations to 4 non-congruent transformations.

Each of the non-congruent transformations is now applied to the system $J$, and the sequences and indices are determined for the five transformed systems.

The 35 triads of the system $JH$ arranged in classes according to their indices are shown in the following table.

| $1^4 26$ | $1^4 35$ | $1^3 2^2 5$ | $1^3 9$ | $1^2 2^3 4$ | $1^2 28$ | $1^2 37$ | $1^2 46$ | $1^2 5^2$ |
|---|---|---|---|---|---|---|---|---|
| 157 | *a*12<br>258 | *a*78<br>*a*56<br>*cg*5 | 168 | *a*34 | *g*14<br>*c*47<br>*dg*8 | *cd*2 | *b*24 | *b*13 |
| $12^2 7$ | 1236 | 1245 | 1 11 | $2^2 4^2$ | $23^2 4$ | 5, 7 | $6^2$ | |
| *ade*<br>*d*46 | *afg*<br>*g*27 | *e*26<br>*f*45<br>*cf*6<br>*c*38<br>*d*37 | *bd*5<br>*bf*8<br>*b*67<br>*beg*<br>*ce*1 | *e*48<br>*df*1<br>*g*36 | *ef*7 | *e*35<br>*f*23 | *abc* | |

The enumeration of the entrances of the elements in the 17 classes shows that the transitive sets of elements are $a$; $b$; $c$; $d$; $e$; $f$; $g$; 1; 2; 3; 4; 5; 6; 7; 8; hence the group for the system is identity. Therefore under the hexagonal transformation the system $J$, with a group of order 4, is changed into a system $JH$ with the group identity. The four quadrangular transformations applied to $J$ yield four non-congruent systems. One of these is a new groupless system $JQ$; the remaining three are known headless systems with groups of orders 2, 2, and 6 respectively.

The work of transforming the 44 known systems has been partially performed and we have discovered about 20 systems of the new groupless type. The headed type of $\Delta_{15}$ is exhausted in 23 systems; the non-headed with a group, in 21. The numerous repetitions now appearing in the work lead us to believe that the groupless type is not much more numerous than the other two types. The purpose of the present investigation is to obtain, if possible, a complete set of groupless systems, and then to show the connecting links between all the systems of the three types.

The system $JH$ with no head and no group, which has been derived by a hexagonal transformation from a system $J$, with no head, but with a group of order 4, may also be derived from another system $IC$ with a head and a group of order 3. Hexagonal transformations leave unchanged the number of $\Delta_7$'s in a $\Delta_{15}$, and, therefore, always transform systems with or without heads into systems with or without heads respectively. A quadrangular transformation may change the number of $\Delta_7$'s in the $\Delta_{15}$. The indices of the new groupless systems show a rather marked difference from the indices of the previously known systems. It has been found convenient to list the indices of systems, so that those indices which contain the greatest number of sequences of period 1 appear at the left-hand side of the index table; that is, the indices are written in descending order of sequences of period 1. Systems with a group contain indices of the types $1^{12}$, $1^8 2^2$,—but no groupless system has yet been found with an index containing more than six sequences of period 1—for example, $1^6 3^2$ contains the maximum number of sequences of period 1 found among the indices of the groupless systems. Hence the indices of the groupless type show a distinct shifting towards the right-hand side of the index table. This shift was to be expected since the new systems are headless and, therefore, the indices necessarily involve fewer sequences of period 1.

VASSAR COLLEGE.

# NOTE ON GREEN'S THEOREM.

BY MR. C. A. EPPERSON.

(Read before the American Mathematical Society April 24, 1915.)

1. *Introduction.*—We wish in this paper to extend Green's theorem to apply to relations equivalent to the general linear partial differential equation of the second order in two variables. The relations are written in such a way as not to involve derivatives of the second order, and the theorem is proved without assuming their existence.* This point of view is desirable for the possibility of its application to physics.

2. *Statement of Lemma.*—We shall denote by $a_{ij}$, $a_i$, $a$, arbitrary continuous functions of $x$ and $y$, the $a_{ij}$ to have also continuous first and second partial derivatives, and the $a_i$ to have continuous first partial derivatives, over a given simply connected region. By $b_{ij}$, $b_i$, $b$, we indicate the coefficients of the linear partial differential expression of the second order which is adjoint to the expression of which the coefficients are $a_{ij}$, $a_i$, $a$. That is, $b_{ij} = a_{ij}$,

$$b_i = 2\frac{\partial a_{i1}}{\partial x} + 2\frac{\partial a_{i2}}{\partial y} - a_i,$$

$$b = \frac{\partial^2 a_{11}}{\partial x^2} + \frac{\partial^2 a_{22}}{\partial y^2} + 2\frac{\partial^2 a_{12}}{\partial x \partial y} - \frac{\partial a_1}{\partial x} - \frac{\partial a_2}{\partial y} + a.$$

* The immediate occasion for the publication of this paper was the appearance of an article by C. W. Oseen, "Über einen Satz von Green und über die Definitionen von Rot und Div," *Rendiconti del Circolo Matematico di Palermo*, vol. 38 (1914), pp. 167–179, which applies this method, by means of the principles of vector analysis, to the special case of Poisson's equation. The general proof is a first step in the detailed working out of a general problem for partial differential equations (see preliminary communication of December, 1913, G. C. Evans, "Green's functions for linear partial differential expressions of the second order, and Green's theorem," BULLETIN, vol. 20, no. 6, March, 1914).

A proof has been given for equations of parabolic type, which may be generalized to the equations here considered. (G. C. Evans, "On the reduction of integro-differential equations," *Transactions Amer. Math. Society*, vol. 15, no. 4, p. 486, Oct., 1914. This paper also gives the literature of the problem). The proof referred to however, by a reduction, makes use of Green's theorem in its usual form; in the present paper, the result is reduced ab initio.

The $b_{ij}$ are then continuous with their first and second partial derivatives, the $b_i$ continuous with their first partial derivatives, and $b$ continuous over the given region. We shall consider a square $S_r$, and by $\lim_{S_r \doteq p}$ we shall mean that no matter how small a circle we take about $p$ the square shall ultimately become and remain entirely within that circle. We have then

LEMMA I.* *Let $u(x, y)$ and $v(x, y)$ and their first partial derivatives be continuous over the given region in the $(xy)$ plane, and let $p$ be a point within that region. If*

$$(f - bu)_p = \lim_{S_r \doteq p} \frac{1}{\sigma_r} \int_{S_r} \left[ \frac{\partial}{\partial x}(b_{11}u) + \frac{\partial}{\partial y}(b_{12}u) - b_1 u \right] dy - \left[ \frac{\partial}{\partial x}(b_{12}u) + \frac{\partial}{\partial y}(b_{22}u) - b_2 u \right] dx \tag{1}$$

*and*

$$(g - av)_p = \lim_{S_r \doteq p} \frac{1}{\sigma_r} \int_{S_r} \left[ \frac{\partial}{\partial x}(a_{11}v) + \frac{\partial}{\partial y}(a_{12}v) - a_1 v \right] dy - \left[ \frac{\partial}{\partial x}(a_{12}v) + \frac{\partial}{\partial y}(a_{22}v) - a_2 v \right] dx; \tag{2}$$

*then*

$$(vf - ug)_p = \lim_{S_r \doteq p} \frac{1}{\sigma_r} \int_{S_r} \left[ a_{11}\left( v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} \right) + a_{12}\left( v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y} \right) + \frac{a_1 - b_1}{2} uv \right] dy - \left[ a_{21}\left( v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} \right) + a_{22}\left( v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y} \right) + \frac{a_2 - b_2}{2} uv \right] dx. \tag{3}$$

3. *Proof of Lemma I.*—If we multiply equation (1) by $v_p$ and equation (2) by $u_p$ and subtract, we have, using the relations existing between the $a$'s and $b$'s of these equations,

* The reader will note that if the second derivatives of $u$ and $v$ exist the theorem corresponds to setting

$$f = a_{11}\frac{\partial^2 u}{\partial x^2} + a_{22}\frac{\partial^2 u}{\partial y^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_1\frac{\partial u}{\partial x} + a_2\frac{\partial u}{\partial y} + au$$

and $g$ = the adjoint of $f$. We then have the ordinary Green's theorem.

$$(vf - ug)_p + \frac{1}{2}\left[\frac{\partial}{\partial x}(a_1 - b_1) + \frac{\partial}{\partial y}(a_2 - b_2)\right]_p u_p v_p$$

$$= \lim_{S_r \doteq p} \frac{1}{\sigma_r} \int_{S_r} \left[ a_{11}\left(v_p \frac{\partial u}{\partial x} - u_p \frac{\partial v}{\partial x}\right) + a_{12}\left(v_p \frac{\partial u}{\partial y} - u_p \frac{\partial v}{\partial y}\right)\right.$$

$$\left. + \frac{a_1 - b_1}{2}(v_p u - u_p v)\right] dy \tag{4}$$

$$- \left[ a_{21}\left(v_p \frac{\partial u}{\partial x} - u_p \frac{\partial v}{\partial x}\right) + a_{22}\left(v_p \frac{\partial u}{\partial y} - u_p \frac{\partial v}{\partial y}\right)\right.$$

$$\left. + \frac{a_2 - b_2}{2}\left(v_p u - u_p v\right)\right] dx.$$

If now we consider the difference function $D$ of the right-hand side of the equations (3) and (4), we have

$$\lim_{S_r \doteq p} D = \lim_{S_r \doteq p} \frac{1}{\sigma_r} \int_{S_r} \left\{ a_{11}\left[(v_p - v_s)\frac{\partial u}{\partial x} - (u_p - u_s)\frac{\partial v}{\partial x}\right]\right.$$

$$+ a_{12}\left[(v_p - v_s)\frac{\partial u}{\partial y} - (u_p - u_s)\frac{\partial v}{\partial y}\right]$$

$$\left. + \frac{a_1 - b_1}{2}[(v_p - v_s)u_s + u_p v_s]\right\} dy$$

$$- \left\{ a_{21}\left[(v_p - v_s)\frac{\partial u}{\partial x} - (u_p - u_s)\frac{\partial v}{\partial x}\right]\right. \tag{5}$$

$$+ a_{22}\left[(v_p - v_s)\frac{\partial u}{\partial y} - (u_p - u_s)\frac{\partial v}{\partial y}\right]$$

$$\left. + \frac{a_2 - b_2}{2}[(v_p - v_s)u_s + u_p v_s]\right\} dx.$$

In order to establish the lemma it is sufficient to show that

$$\lim_{S_r \doteq p} D = \frac{1}{2}\left[\frac{\partial}{\partial x}(a_1 - b_1) + \frac{\partial}{\partial y}(a_2 - b_2)\right]_p u_p v_p.$$

To obtain the value of $D$ we will divide the square into four smaller rectangles by lines through $p$ parallel to the sides of the square and consider the sum of the integrals extended around these smaller rectangles. For convenience of notation

we shall consider $p$ as the origin, and the vertices of the square lettered as in accompanying figure (Fig. 1). We will write

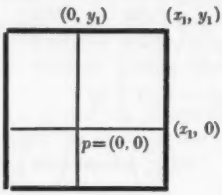


Fig. 1.

the curvilinear integral $D_1$ around one of these small rectangles, say $[(0, 0)/(x_1, y_1)]$, as a simple integral.

Thus

$$
\begin{aligned}
D_1 = \frac{1}{\sigma_r}\int_0^{x_1}\Bigg[\Bigg\{ & a_{11}\left[(v_p - v_s)\frac{\partial u}{\partial x} - (u_p - u_s)\frac{\partial v}{\partial x}\right] \\
& + a_{12}\left[(v_p - v_s)\frac{\partial u}{\partial y} - (u_p - u_s)\frac{\partial v}{\partial y}\right] \\
& + \frac{a_1 - b_1}{2}\left[(v_p - v_s)u_s + u_p v_s\right]\Bigg\}_{x_1, y} \\
& - \Bigg\{ a_{11}\left[(v_p - v_s)\frac{\partial u}{\partial x} - (u_p - u_s)\frac{\partial v}{\partial x}\right] \\
& + a_{12}\left[(v_p - v_s)\frac{\partial u}{\partial y} - (u_p - u_s)\frac{\partial v}{\partial y}\right] \\
& + \frac{a_1 - b_1}{2}\left[(v_p - v_s)u_s + u_p v_s\right]\Bigg\}_{0, y}\Bigg] dy
\end{aligned}
\tag{6}
$$

plus the corresponding term to be integrated with respect to $x$.

If we note the figure we see that

$$
\left|(v_s - v_p) - \left(\frac{\partial v_s}{\partial x}x_1 + \frac{\partial v_s}{\partial y}y\right)\right| < \epsilon(|x_1| + |y|) < 2\epsilon\delta
$$

$0 \leq y \leq y_1$, $0 \leq x \leq x_1$, $\delta$ = one side of the square,

if $v_s$ has the coordinates $(x_1, y)$, and a similar expression holds if $v_s$ is on one of the other sides. Similar expressions hold

for $(u_s - u_p)$. Further, one infinitesimal $\epsilon$ may be used for all such expressions, as $u$ and $v$ and their first partial derivatives are uniformly continuous throughout the square. In the limit then $(v_s - v_p)$ may be replaced by $(\partial v_s/\partial x)x_1 + (\partial v_s/\partial y)y$ or by the corresponding expressions depending on the position of $v_s$.

If we carry out these substitutions in detail we obtain an expression of the form

$$
\begin{aligned}
D_1 = \frac{1}{\sigma_r}\int_o^{y_1}\Bigg[ & \left\{\left[a_{11}\left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\right]_{x_1, y}\right. \\
& \left. - \left[a_{11}\left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\right]_{0, y}\right\}(-y) \\
& + \left\{\left[a_{12}\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)\right]_{x_1, y}\right. \\
& \left. - \left[a_{12}\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)\right]_{0, y}\right\}(-x_1) \\
& + \left\{\left[\frac{a_1 - b_1}{2}\left(\frac{\partial v}{\partial y}u\right)\right]_{x_1, y}\right. \\
& \left. - \left[\frac{a_1 - b_1}{2}\left(\frac{\partial v}{\partial y}u\right)\right]_{0, y}\right\}(-y) \\
& + u_p\left[\left(\frac{a_1 - b_1}{2}v\right)_{x_1, y} - \left(\frac{a_1 - b_1}{2}v\right)_{0, y}\right] \\
& + \left(\frac{a_1 - b_1}{2}\frac{\partial v}{\partial x}u\right)_{x_1, y}(-x_1)\Bigg]dy
\end{aligned}
\tag{7}
$$

plus exactly similar terms to be integrated with respect to $x$, plus terms every one of which is in absolute value less than $\epsilon$ times a constant less than $|\sigma_r|$.

If we apply the law of the mean every term, except the last two, is of one of the forms

$$(a)\quad [P(x_1, y) - P(0, y)]\left(\frac{x_1 y_1}{\sigma_r}\right) \text{ or } [P(x_1, y) - P(0, y)]\left(\frac{x_1 x_1}{\sigma_r}\right)$$

$$(b)\qquad \epsilon\left(\frac{x_1 y_1}{\sigma_r}\right) \text{ or } \epsilon\left(\frac{x_1 x_1}{\sigma_r}\right), \text{etc.},$$

where $P(x, y)$ is uniformly continuous in $x$ and in $y$ throughout the square. Hence in the limits every term except the last two vanishes and we have

$$D_1 = \frac{1}{\sigma_r}\left[\left(\frac{a_1 - b_1}{2}\frac{\partial v}{\partial x}u\right)_{x_1, y}(-x_1) + u_p\frac{\partial}{\partial x}\left(\frac{a_1 - b_1}{2}v\right)_{0, y}(x_1)\right]y_1 + \eta, \tag{8}$$

where $\lim_{S_r \doteq p} \eta = 0$,

$$= \frac{1}{\sigma_r}u_p\left[\left(\frac{a_1 - b_1}{2}\frac{\partial v}{\partial x}\right)_{x_1, y}(-x_1) + \left(\frac{a_1 - b_1}{2}\frac{\partial v}{\partial x}\right)_{0, y}(x_1) + \left(v\frac{\partial}{\partial x}\frac{a_1 - b_1}{2}\right)_{0, y}(x_1)\right]y_1 + \eta + \eta',$$

where

$$|\eta'| < \epsilon\left|\frac{a_1 - b_1}{2}\frac{\partial v}{\partial x}\right|,$$

which gives us, if we combine with it the corresponding terms from the integration with respect to $x$,

$$D_1 = \left(\frac{x_1 y_1}{\sigma_r}\right)\frac{1}{2}\left[\frac{\partial}{\partial x}(a_1 - b_1) + \frac{\partial}{\partial y}(a_2 - b_2)\right]_p u_p v_p + \omega,$$

where $\lim_{S_r \doteq p} \omega = 0$, $\omega$ being less in absolute value than $\epsilon\delta^2/\sigma_r$ times some constant.

If we now carry the integration around the other rectangles we have similar expressions for $D_2$, $D_3$, and $D_4$.

Since $(x_1y_1 + x_2y_1 + x_2y_2 + x_1y_2)/\sigma_r = 1$, we have

$$\lim_{S_r \doteq p} D = \frac{1}{2}\left[\frac{\partial}{\partial x}(a_1 - b_1) + \frac{\partial}{\partial y}(a_2 - b_2)\right]_p u_p v_p.$$

The lemma is therefore established.

4. *Statement of Green's Theorem.*—By a *standard curve* we shall mean a closed regular curve which does not cut itself and which cannot be cut in more than a finite number of points by any horizontal or vertical straight line.

We may now state Green's theorem.

THEOREM I. *If $u(x, y)$ and $v(x, y)$ with their first partial derivatives, and $f$ and $g$, are continuous throughout a simply*

*connected region, and if for every standard curve S,* contained wholly within that region,*

$$\iint_\sigma (f - bu)dxdy = \int_s \left[\frac{\partial}{\partial x}(b_{11}u) + \frac{\partial}{\partial y}(b_{12}u) - b_1 u\right]dy - \left[\frac{\partial}{\partial x}(b_{21}u) + \frac{\partial}{\partial y}(b_{22}u) - b_2 u\right]dx \tag{9}$$

*and*

$$\iint_\sigma (g - av)dxdy = \int_s \left[\frac{\partial}{\partial x}(a_{11}v) + \frac{\partial}{\partial y}(a_{12}v) - a_1 v)\right]dy - \left[\frac{\partial}{\partial x}(a_{21}v) + \frac{\partial}{\partial y}(a_{22}v) - a_2 v\right]dx; \tag{10}$$

*then*

$$\begin{aligned}\iint_\sigma (vf - ug)dxdy = \int_s \Big[ & a_{11}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) \\ & + a_{12}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \frac{a_1 - b_1}{2}uv\Big]dy \\ & - \Big[a_{21}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + a_{22}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) \\ & + \frac{a_2 - b_2}{2}uv\Big]dx.\end{aligned} \tag{11}$$

*Proof of Theorem I.*—For convenience in writing our equations we will set

$$A_i = a_{i1}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + a_{i2}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \frac{a_i - b_i}{2}uv.$$

Now divide the region enclosed by the contour into a number of squares $\sigma_i$ of side $\delta$. Then

$$\iint_\sigma (vf - ug)dxdy = \sum^n \iint_{\sigma_i} (vf - ug)dx_i dy_i + \eta \quad (n > N),$$

where $\eta$ is the integral over that portion of $\sigma$ not contained in $\sum^n \sigma_i$ and can be made as small as we please by choosing $n$ sufficiently large. By the law of the mean

* The conclusion holds for the boundary of the region if it is a standard curve.

$$\int\int_{\sigma_i}(vf - ug)dx_i dy_i = (vf - ug)_{p_i}\sigma_i,$$

where $p_i$ is some point in $\sigma_i$. Then

$$\int\int_{\sigma}(vf - ug)dxdy = \sum^{n}(vf - ug)_{p_i}\sigma_i + \eta \quad (n > N).$$

The hypotheses of this theorem imply the hypotheses of Lemma I. Hence, by that lemma,

$$(vf - ug)_{p_i} = \lim_{S_i \doteq p_i} \frac{1}{\sigma_i}\int_{S_i} A_1 dy - A_2 dx,$$

so that

$$\int\int_{\sigma}(vf - ug)dxdy = \sum^{n}(vf - ug)_{p_i}\sigma_i + \eta$$

$$= \sum^{n}\sigma_i \lim_{S_i \doteq p_i} \frac{1}{\sigma_i}\int_{S_i} A_1 dy - A_2 dx. \tag{12}$$

We have to do with a uniform limit in the right-hand member of the above equation, since on account of the assumed continuity of $u$, $\partial u/\partial x$, $\partial u/\partial y$ the quantity $\epsilon$ which occurs in the proof of Lemma I depends merely on the size of the square, and not on the position of the point $p$. Hence

$$\lim_{S_i \doteq p_i} \frac{1}{\sigma_i}\int_{S_i} A_1 dy - A_2 dx$$

differs from

$$\frac{1}{\sigma_i}\int_{S_i} A_1 dy - A_2 dx$$

by less than $\bar{\epsilon}$, where $\lim_{\delta \doteq 0} \bar{\epsilon} = 0$, and

$$\sum^{n}\sigma_i \lim_{S_i \doteq p_i} \frac{1}{\sigma_i}\int_{S_i} A_1 dy - A_2 dx$$

differs from

$$\int_{S_r} A_1 dy - A_2 dx$$

by less than $\bar{\epsilon}\sigma$, where $S_r$ is the contour formed by the exterior boundary of the approximating squares. It then follows that

$$\int\int_{\sigma}(vf - ug)dxdy = \int_{S_r} A_1 dy - A_2 dx + \eta',$$

where $\lim_{\delta \doteq 0} \eta' = 0$. But since $A_1$ and $A_2$ are uniformly continuous in $x$ and $y$ throughout the region, we have

$$\lim_{\delta \doteq 0} \int_{S_\sigma} A_1 dy - A_2 dx + \eta' = \int_S A_1 dy - A_2 dx,$$

and therefore finally

$$\int\int_\sigma (vf - ug) dx dy = \int_S A_1 dy - A_2 dx,$$

which was to be proved.

COROLLARY I. *Theorem for Poisson's Equation. If $u$ and $v$ and their first partial derivatives, and $f$ and $g$, are continuous throughout a simply connected region, and if for every standard curve $S$ contained wholly within that region,*

$$\int\int_\sigma f(x, y) dx dy = \int_S \frac{\partial u}{\partial n} ds$$

*and*

$$\int\int_\sigma g(x, y) dx dy = \int_S \frac{\partial v}{\partial n} ds$$

*then*

$$\int\int_\sigma (vf - ug) dx dy = \int_S \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds.$$

If we put in this $v = 1/r$ we get the well-known formula due to Green.

COROLLARY II. By $\lim_{S \doteq p}$ we shall mean that the standard curve $S$ approaches $p$ uniformly, i. e., if we take a circle with center at $p$ and of arbitrarily small radius the standard curve $S$ ultimately becomes and remains within the circle. We may now state

COROLLARY II. *If $u$ and $v$ and their first partial derivatives are continuous within and on the boundary of a region enclosed by a standard curve $S$ in the $(xy)$ plane, and if*

$$(f - bu)_p = \lim_{S \doteq p} \frac{1}{\sigma} \int_S \left[ \frac{\partial}{\partial x} (b_{11} u) + \frac{\partial}{\partial y} (b_{22} u) - b_2 u \right] dy - \left[ \frac{\partial}{\partial x} (b_{12} u) + \frac{\partial}{\partial y} (b_{22} u) - b_2 u \right] dx$$

*and*

$$(g - av)_p = \lim_{S \to p} \frac{1}{\sigma} \int_s \left[ \frac{\partial}{\partial x}(a_{11}v) + \frac{\partial}{\partial y}(a_{12}v) - a_1 v \right] dy - \left[ \frac{\partial}{\partial x}(a_{21}v) + \frac{\partial}{\partial y}(a_{22}v) - a_2 v \right] dx,$$

*then*

$$(vf - ug)_p = \lim_{S \to p} \frac{1}{\sigma} \int_s \left[ a_{11}\left( v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} \right) + a_{12}\left( v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y} \right) + \frac{a_1 - b_1}{2} uv \right] dy - \left[ a_{21}\left( v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} \right) + a_{22}\left( v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y} \right) + \frac{a_2 - b_2}{2} uv \right] dx,$$

*where $p$ is any point within $S$.*

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# CONVERGENCE OF THE SERIES $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i y^j}{i - j\gamma}$ ($\gamma$ IRRATIONAL).

BY PROFESSOR W. D. MACMILLAN.

(Read before the American Mathematical Society April 2, 1915.)

The method of proof which is here used depends upon the properties of continued fractions. Any irrational number $\gamma$ can be expanded as a simple continued fraction

$$\gamma = a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4 +} \cdots.$$

Let $p_n/q_n$ be the $n$th principal convergent,* and $P/Q$ be any intermediate convergent lying between $p_{n-2}/q_{n-2}$ and $p_n/q_n$. Then

$$\frac{p_{n-2}}{q_{n-2}} < \frac{P}{Q} < \frac{p_n}{q_n} < \gamma < \frac{p_{n+1}}{q_{n+1}} < \frac{p_{n-1}}{q_{n-1}}$$

if $n$ is odd, and

* The notation used here agrees with that of Chrystal's Algebra, Vol. II, Chap. XXXII.

$$\frac{p_{n-2}}{q_{n-2}} > \frac{P}{Q} > \frac{p_n}{q_n} > \gamma > \frac{p_{n+1}}{q_{n+1}} > \frac{p_{n-1}}{q_{n-1}}$$

if $n$ is even.

For $n$ either even or odd

$$\left|\frac{P}{Q} - \gamma\right| > \left|\frac{P}{Q} - \frac{p_n}{q_n}\right| = \left|\frac{p_{n-2} + kp_{n-1}}{q_{n-2} + kq_{n-1}} - \frac{p_n}{q_n}\right|$$

$$1 \leqq k \leqq (a_n - 1),$$

from which is derived

$$\left|\frac{P}{Q} - \gamma\right| > \frac{a_n - k}{q_n Q}.$$

Since $q_n = q_{n-2} + a_n q_{n-1}$ and $Q = q_{n-2} + kq_{n-1}$, we have

$$q_n = Q + (a_n - k)q_{n-1} < Q + a_n q_{n-1} < (a_n + 1)Q.$$

Hence

$$\left|\frac{P}{Q} - \gamma\right| > \frac{a_n - k}{q_n Q} \geqq \frac{1}{q_n Q} > \frac{1}{Q^2(a_n + 2)},$$

and likewise

$$\left|\frac{p_n}{q_n} - \gamma\right| > \frac{1}{q_n(q_{n+1} + q_n)} > \frac{1}{q_n^2(a_{n+1} + 2)}.$$

We have then

$$|P - Q\gamma| > \frac{1}{Q(a_n + 2)} \tag{1}$$

and

$$|p_{n-1} - q_{n-1}\gamma| > \frac{1}{q_{n-1}(a_n + 2)}.$$

Let us suppose now that $a_n + 2 < M$ for every $n$, an hypothesis which is certainly satisfied by every simple quadratic surd $(m \pm \sqrt{n})/l$, where $l$, $m$ and $n$ are integers. Then if $P/Q$ is any convergent, principal or intermediate, it follows from (1) that

$$|P - Q\gamma| > \frac{1}{MQ}. \tag{2}$$

We shall show now that for any two integers whatever, $i$ and $j$,

$$|i - j\gamma| > \frac{1}{Mj}. \tag{3}$$

Let $P_1/Q_1$ and $P_2/Q_2$ be two successive convergents (principal or intermediate) such that $Q_1 \leqq j \leqq Q_2$. From the general theory of continued fractions it is known that $P_1/Q_1$ and $P_2/Q_2$ are closer approximations to $\gamma$ than any other rational fractions whose denominators are less than $Q_2$. Consequently

$$\left|\frac{i}{j} - \gamma\right| > \left|\frac{P_1}{Q_1} - \gamma\right| > \frac{1}{MQ_1^2},$$

and therefore, since $j > Q_1$,

$$|i - j\gamma| > |P_1 - Q_1\gamma| > \frac{1}{MQ_1} > \frac{1}{Mj}. \tag{4}$$

Consider now the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^i y^j}{i - j\gamma} \qquad (i + j > 0),$$

where $\gamma$ is an irrational number which, when expressed as a simple continued fraction, satisfies the condition that $a_n + 2 < M$ for every $n$. Then we will have $|i - j\gamma| > 1/(Mj)$ and consequently

$$\frac{1}{|i - j\gamma|} < Mj;$$

so that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^i y^j}{i - j\gamma} = \sum_{i=1}^{\infty}\frac{x^i}{i} + \sum_{i=0}^{\infty}x^i\sum_{j=1}^{\infty}\frac{y^j}{i - j\gamma} \ll \sum_{i=1}^{\infty}x^i$$

$$+ M\sum_{i=1}^{\infty}x^i\sum_{j=1}^{\infty}jy^j \ll \frac{x}{1-x} + \frac{M}{1-x}\cdot\frac{y}{(1-y)^2},$$

which converges if $|x| < 1$ and $|y| < 1$. Therefore the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^i y^j}{i - j\gamma} \qquad (i + j > 0)$$

converges if $|x|$ and $|y|$ are both less than unity, which is somewhat remarkable in that the denominators, which have no lower limit, impose no restriction upon the radii of convergence of the series.

The restrictions upon $\gamma$ in the above discussion can be considerably reduced. It is seen from (1), if $P/Q$ is any convergent (principal or intermediate), that

$$|P - Q\gamma| > \frac{1}{Q(a_n + 2)},$$

where $a_n$ is the first partial quotient above $Q$. Let us suppose now that

$$(5) \quad a_n + 2 < M(q_{n-1} + 1)(q_{n-1} + 2)\cdots(q_{n-1} + S - 1),$$

where $S$ is any positive integer independent of $n$. Then, since $Q > q_{n-1}$,

$$a_n + 2 < M(Q + 1)\cdots(Q + S - 1),$$

so that

$$|P - Q\gamma| > \frac{1}{MQ(Q + 1)\cdots(Q + S - 1)}.$$

Then, just as before, if $i$ and $j$ are any two integers such that $Q \leqq j \leqq Q_1$, we shall have

$$|i - j\gamma| > |P - Q\gamma| > \frac{1}{MQ(Q + 1)\cdots(Q + S - 1)}$$

$$> \frac{1}{Mj(j + 1)\cdots(j + S - 1)}$$

and also

$$\frac{1}{|i - j\gamma|} < Mj(j + 1)\cdots(j + S - 1).$$

If then $\gamma$ satisfies these new conditions the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^i y^j}{i - j\gamma} \ll \frac{x}{1 - x} + M\sum_{i=0}^{\infty} x^i y \sum_{j=1}^{\infty} j(j+1)\cdots(j+S-1)y^{j-1}.$$

But since

$$\sum_{j=1}^{\infty} j(j + 1)\cdots(j + S - 1)y^{j-1} = \frac{d^S}{dy^S}\left(\frac{1}{1 - y}\right) = \frac{S!}{(1 - y)^{S+1}}$$

we have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^i y^j}{i - j\gamma} \ll \frac{x}{1 - x} + \frac{yMS!}{(1 - x)(1 - y)^{S+1}}$$

and therefore convergent provided $|x|$ and $|y|$ are both less than unity.

*Corollary.*—If the series $f(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}x^iy^j \quad (i+j>0)$ converges for $|x| < 1/\xi$, $|y| < 1/\eta$, so that

$$f(x, y) \ll \frac{N}{(1-\xi x)(1-\eta y)} - N,$$

and if $\gamma$ is an irrational number which satisfies (5), then the series

$$F(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{a_{ij}}{i-j\gamma} x^iy^j \qquad (i+j>0),$$

converges provided $|x| < 1/\xi$ and $|y| < 1/\eta$. Furthermore

$$F(x, y) \ll \frac{N}{1-\xi x}\left[\xi x + \frac{MS!}{(1-\xi x)(1-\eta y)^{S+1}}\right].$$

It will perhaps be interesting to note the character of the condition that $a_{n+1} + 2 < Mq_n(q_n+1)\cdots(q_n+S-1)$. Let us suppose that $a_n = n!$. It is found then $q_{n-1} = (n-1)!(n-2)!\cdots 2! + \cdots$. It is sufficient then to take $M = 1$, $S = 2$, in order to satisfy the condition. If we suppose that $a_n = 10^{10^{n-1}}$ we find that $q_{n-1} = 10^{10^{n-2}}\cdot 10^{10^{n-3}}\cdot \cdots 10^{10^0} + \cdots$, and it is sufficient to take $M = 10$, $S = 10$. If however we suppose that $a_n = 10^{n!}$ then $q_{n-1} = 10^{(n-1)!+(n-2)!+\cdots+2!} + \cdots$, and there do not exist an $M$ and an $S$ which satisfy the condition.

*Application of these Series.*—(*a*) The function

$$W = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\lambda^m\mu^n}{m-nz}, \quad |\lambda| < 1, \quad |\mu| < 1,$$

where $z$ is a complex variable, is a holomorphic function of $z$ everywhere except in the neighborhood of the positive real axis, which is a line of essential singularities. Nevertheless the value of the function is finite for those real, positive irrational values of $z$ which satisfy the above condition; furthermore the function is continuous across the real axis at any one of these points. To show the continuity, let $z = \gamma$ be such a point and let $z = \gamma + t\cos\alpha + it\sin\alpha$ be a

straight line which crosses the real axis at $\gamma$ making an angle $\alpha$ with the real axis. Then

$$W = \Sigma\Sigma \frac{\lambda^m \mu^n}{(m - n\gamma - nt\cos\alpha) - i(nt\sin\alpha)}$$

$$= \Sigma\Sigma \frac{[m - n\gamma - nt\cos\alpha] + i(nt\sin\alpha)}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha} \lambda^m\mu^n.$$

If now we write $W = W_1 + iW_2$, and for brevity suppose $\lambda$ and $\mu$ real, we have

$$W_1 = \Sigma\Sigma\lambda^m\mu^n \frac{m - n\gamma - nt\cos\alpha}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha},$$

$$W_2 = \Sigma\Sigma\lambda^m\mu^n \frac{nt\sin\alpha}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha}.$$

Consider now

$$\frac{nt\sin\alpha}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha}.$$

As a function of the variable $t$ this expression has a maximum or a minimum for $n^2t^2 = (m - n\gamma)^2$. It has a maximum equal to

$$\frac{1}{(m - n\gamma)[(1 + \cos\alpha)^2 + \sin^2\alpha]}$$

for $nt = (m - n\gamma)$, and a minimum equal to

$$\frac{-1}{(m - n\gamma)[(1 + \cos\alpha)^2 + \sin^2\alpha]}$$

for $nt = -(m - n\gamma)$. Consequently

$$W_2 \ll \frac{|\sin\alpha|}{[(1 - \cos\alpha)^2 + \sin^2\alpha]} \cdot \Sigma\Sigma \frac{\lambda^m\mu^n}{|m - n\gamma|},$$

which is absolutely convergent. Whence $W_2$, and in the same manner $W_1$, is absolutely and uniformly convergent for all real values of $t$. Consequently $W$ is a continuous function of $z$ all along this straight line.

(*b*) Consider the linear partial differential equation

$$x_1\frac{\partial\phi}{\partial x_1} - \gamma x_2\frac{\partial\phi}{\partial x_2} = p_1\phi + p_2,$$

where $\gamma$ is a positive irrational number which satisfies the condition $a_{n+1} + 2 < Mq_n(q_n + 1)\cdots(q_n + S - 1)$, and $p_1 = \Sigma\Sigma a_{ij}x_1^i x_2^j$, $p_2 = \Sigma\Sigma b_{ij}x_1^i x_2^j$, are two convergent power series in $x_1$ and $x_2$.

We will take first the homogeneous equation

$$x_1\frac{\partial\phi}{\partial x_1} - \gamma x_2\frac{\partial\phi}{\partial x_2} = p_1\phi,$$

and put $\psi = \log\phi$. Then

$$x_1\frac{\partial\psi}{\partial x_1} - \gamma x_2\frac{\partial\psi}{\partial x_2} = p_1.$$

The solution of this equation is

$$\psi = \Sigma\Sigma\frac{a_{ij}}{i - j\gamma}x_1^i x_2^j + \text{an arbitrary function of } (x_1^\gamma x_2),$$

and by the above corollary this series has the same region of validity as $p_1$ itself. It follows therefore that $\phi = e^\psi$ also is a convergent power series in $x_1$ and $x_2$, if the arbitrary function is taken equal to zero.

Returning now to the equation

$$x_1\frac{\partial\phi}{\partial x_1} - \gamma x_2\frac{\partial\phi}{\partial x_2} = p_1\phi + p_2,$$

let us take $\phi = \omega e^\psi$, where $e^\psi$ is the function already determined, and $\omega$ is an unknown function. We have then

$$x_1\frac{\partial\omega}{\partial x_1} - \gamma x_2\frac{\partial\omega}{\partial x_2} = p_2 e^{-\psi} = \Sigma\Sigma c_{ij}x_1^i x_2^j,$$

where $\Sigma\Sigma c_{ij}x_1^i x_2^j$ is the expansion of $p_2 e^{-\psi}$ and is therefore a convergent series. The solution of this equation is

$$\omega = \Sigma\Sigma\frac{c_{ij}}{i - j\gamma}x_1^i x_2^j + \text{an arbitrary function of } (x_1^\gamma x_2),$$

which likewise is a convergent series. Consequently $\phi = (A + \omega)e^\psi$, where $A$ is an arbitrary function of $(x_1^\gamma x_2)$, is a solution of the differential equation.

THE UNIVERSITY OF CHICAGO,
*May* 15, 1915.

# A CERTAIN CLASS OF FUNCTIONS CONNECTED WITH FUCHSIAN GROUPS.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society April 24, 1915.)

1. CONSIDER a Fuchsian group $\Gamma$ of linear substitutions

$$(1) \qquad V_i \equiv z_i = \frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} \qquad (i = 1, 2, 3, \cdots)$$
$$\alpha_i\delta_i - \beta_i\gamma_i = 1,$$

that transform the unit circle into itself, and for which the unit circle is a natural boundary. The index $i$ for which $z_i$ approaches a point of the boundary we denote by $\infty$, so that $\lim_{i=\infty}(z_i) = e^{i\phi}$, where $\phi$ may have any value from 0 to $2\pi$. Let $z_0 = z$ represent identity. Denote by $R_0 = R$ the fundamental region in which $z$ lies, and by $R_1, R_2, \cdots$ the regions resulting from $R$ by the substitutions $V_i$ $(i = 1, 2, 3, \cdots)$. Let $e_i$ be the greatest "elongation" of the boundary of $R_i$, i. e., the maximum distance between two points of the boundary of $R_i$; then, according to a theorem due to Bricard,* it is possible to circumscribe a circle $C_i$ to the region $R_i$, such that its radius does not need to be greater than at most $e_i/\sqrt{3}$.

For $i \neq \infty$, the area $A_i$ of $R_i$, being that of a singly connected region bounded by circular arcs, is finite, so that for the ratio of the area of the circle $C_i$ to that of the region $R_i$ we have

$$(2) \qquad 1 < \frac{\pi e_i^2}{3A_i} < M \qquad (i = 1, 2, 3, \cdots),$$

where $M$ is a positive finite quantity $> 1$. But it can be shown that this inequality also exists when $\lim_{=\infty}(i) = \infty$, or $\lim_{=\infty}(z_i) = e^{i\phi}$. Hence from (2) we get

$$(3) \qquad 3\Sigma A_i < \Sigma\pi e_i^2 < 3M\Sigma A_i,$$

* "Théorèmes sur les courbes et les surfaces fermées," *Nouvelles Annales de Mathématique*, 4. ser., vol. 14, pp. 19–25 (January, 1914).

in which the sums are extended over the whole group $\Gamma$. As $\Sigma A_i = \pi$ is a finite quantity we find that the sum of the areas of all circles $C_i$, and consequently the sum of the squares of the radii of all these circles is finite.

2. Choose now within $R$ any two points $a$ and $b$ and a variable point $z$, so that the area formed by the euclidean triangle $z_i a_i b_i$ lies entirely within $C_i$. Now

$$|z_i - a_i| \leqq e_i;\ |z_i - b_i| \leqq e_i,$$

hence

$$|z_i - a_i| \cdot |z_i - b_i| \leqq e_i^2,$$

and

$$\sum_{i=0}^{\infty} |z_i - a_i| \cdot |z_i - b_i| \leqq \sum_{i=0}^{\infty} e_i^2. \tag{4}$$

But, according to (3), $\sum_{i=0}^{\infty} e_i^2$ is a finite quantity. The left side of (4) is therefore an absolutely convergent series, for all values of $z$ within $R$. The condition for uniform convergence within the whole domain is evidently also satisfied, so that we can state

THEOREM I. *The series*

$$\sum_{i=0}^{\infty} (z_i - a_i)(z_i - b_i)$$

*extended over a Fuchsian group* $\Gamma$, *with the unit circle as a natural boundary and* $z$, $a$, $b$ *lying within the fundamental region of* $\Gamma$, *is a uniformly convergent series, and defines an analytic function within* $R$ *that vanishes for* $z = a$ *and* $z = b$ *and has no infinities within* $R$. *The result is still valid when* $z_b = z_a$, *so that*

$$\sum_{i=0}^{\infty} (z_i - a_i)^2$$

*also defines such a function which at* $z = a$ *has a zero of the second order.*

3. The theorem may immediately be generalized. Choose for $z$ and $a$ any two points within the unit circle (excluding the boundary). The straight line joining them is cut by a finite number of polygons $R_i$ into the segments $l_1, l_2, l_3, \cdots, l_r$. Any substitution $V_\lambda \equiv \begin{pmatrix} \alpha_\lambda \beta_\lambda \\ \gamma_\lambda \delta_\lambda \end{pmatrix}$ of the group $\Gamma$ transforms the

straight segment from $z$ to $a$ into an arc of a circle from $z_\lambda$ to $a_\lambda$ and the segments $l_i$ into arcs $l_{i\lambda}$ intercepted by the corresponding polygons arising from the substitution $V_\lambda$. Every arc $l_{i\lambda}$ is subtended by a chord $s_{i\lambda} < l_{i\lambda} < e_{i\lambda}$, where $e_{i\lambda}$ denotes the elongation of the polygon (region) $R_{i\lambda}$. From this follows immediately that

$$f_\lambda = |z_\lambda - a_\lambda| < e_{1\lambda} + e_{2\lambda} + \cdots + e_{r\lambda},$$

and

$$(f_\lambda)^2 < (e_{1\lambda} + e_{2\lambda} + \cdots + e_{i\lambda} + \cdots + e_{k\lambda} + \cdots + e_{r\lambda})^2.$$

From the inequality

$$2e_{i\lambda}e_{k\lambda} < e_{i\lambda}^2 + e_{k\lambda}^2$$

we derive without difficulty

$$(5) \qquad 2\sum_{\substack{i=1,\ k=1 \\ i\neq k}}^{r} e_{i\lambda}e_{k\lambda} < (r-1)(e_{1\lambda}^2 + e_{2\lambda}^2 + \cdots + e_{r\lambda}^2).$$

Now

$$\sum_{\lambda=0}^{\infty}(f_\lambda)^2 = \sum_{\lambda=0}^{\infty}(e_{1\lambda}^2 + e_{2\lambda}^2 + \cdots + e_{r\lambda}^2) + 2\sum_{\lambda=0}^{\infty}\sum_{\substack{i=1,\ k=1 \\ i\neq k}}^{r} e_{i\lambda}e_{k\lambda};$$

hence, according to (5),

$$(6) \qquad \sum_{\lambda=0}^{\infty}(f_\lambda)^2 < r\sum_{\lambda=0}^{\infty}(e_{1\lambda}^2 + e_{2\lambda}^2 + \cdots + e_{r\lambda}^2).$$

But

$$\sum_{\lambda=0}^{\infty} e_{i\lambda}^2 = \sum_{\lambda=0}^{\infty} e_{k\lambda}^2,$$

so that (6) reduces to

$$(7) \qquad \sum_{\lambda=0}^{\infty}(f_\lambda)^2 < r^2\sum_{\lambda=0}^{\infty} e_\lambda^2.$$

The right side of this inequality is a finite quantity, so that the series on the left side is absolutely convergent. Hence

THEOREM II. *The series*

$$\sum_{\lambda=0}^{\infty}(z_\lambda - a_\lambda)^2$$

*extended over a Fuchsian group with the unit circle as a natural boundary, where z and a are any two points within the unit circle and not on the boundary, when a is fixed, is an absolutely and uniformly convergent series of z for all points within and not on the boundary, and represents an analytic function in the neighborhood of all such points. It has a zero of the second order for $z = a$, and has the unit circle as a natural boundary.*

4. This theorem admits of a further generalization. Choose any three points $z$, $z'$, $a$ within and not on the unit circle, and write $f_\lambda = |z_\lambda - a_\lambda|$, $g_\lambda = |z_\lambda' - a_\lambda|$. Assuming $f \neq 0$ $g \neq 0$, it is possible to find a positive finite number $M$ such that the ratio $g_\lambda/f_\lambda < M$, $\lambda = 1, 2, 3, \cdots$, also when $z_\lambda$ approaches a point on the unit circle. We have therefore $g_\lambda < Mf_\lambda$, and

$$f_\lambda g_\lambda < Mf_\lambda^2,$$

and consequently

$$\sum_{\lambda=0}^{\infty} f_\lambda g_\lambda < M \sum_{\lambda=0}^{\infty} f_\lambda^2. \tag{8}$$

As the right side of this inequality is absolutely convergent, it follows that

$$\sum_{\lambda=0}^{\infty} f_\lambda g_\lambda$$

is an absolutely convergent series, and that consequently

$$\sum_{\lambda=0}^{\infty} (z_\lambda - a_\lambda)(z_\lambda' - a_\lambda) \tag{9}$$

is absolutely and uniformly convergent, and, for $a$ and $z'$ constant, defines an analytic function of $z$ for all points within and not on the boundary of the unit circle. It vanishes for $z = a$ and has the unit circle as a natural boundary. Nothing is lost in the convergency proof of (9) by assuming $z$ and $z'$ fixed and $a$ as variable. Hence putting in (9) $z = a$, $z' = b$ and $a = z$ we may state

THEOREM III. *The series*

$$\sum_{\lambda=0}^{\infty} (z_\lambda - a_\lambda)(z_\lambda - b_\lambda),$$

*where a and b are any two points within and not on the unit circle, is absolutely and uniformly convergent and represents an analytic*

*function of $z$ within the unit circle, which is a natural boundary of the function. It has $z = a$ and $z = b$ as zeros.*

5. Making use of the proposition that for an analytic function $F(z)$ which within a certain region has the character of a rational function, such that for any point $z_0$ of this region $F(z_0)$ exists,

$$\lim_{z=z'=z_0} \left( \frac{F(z) - F(z')}{z - z'} \right) = F'(z_0) \tag{10}$$

we may extend theorem III to an even more general type of functions. Let $\mathfrak{R}(z)$ be a rational function of $z$ which for $z = 0$ does not become infinite. Putting $(z_\lambda - a_\lambda)(z_\lambda - b_\lambda) = u_\lambda$, $(z_\lambda' - a_\lambda')(z_\lambda' - b_\lambda') = u_\lambda'$, where $z'$, $a'$, $b'$ denote a set like $z$, $a$, $b$, then as $z_\lambda$ $(\lambda = \infty)$ approaches a definite point $e^{i\phi}$ on the boundary $a_\lambda$, $b_\lambda$, $a_\lambda'$, $b_\lambda'$, $z_\lambda'$ will approach the same point, and $u$ and $u'$ will approach zero as a limit. Consequently

$$\lim_{z_\lambda = e^{i\phi}} \left\{ \frac{\mathfrak{R}(u_\lambda) - \mathfrak{R}(u_\lambda')}{u_\lambda - u_\lambda'} \right\} = \mathfrak{R}'(0) \tag{11}$$

is a finite quantity, and as $\Sigma(u - u')$ is absolutely and uniformly convergent, also $\sum_{\lambda=0}^{\infty} \{\mathfrak{R}(u) - \mathfrak{R}(u')\}$ will be absolutely and uniformly convergent within the unit circle, except for a finite number of values of $u$ and $u'$, which are poles of $\mathfrak{R}(u)$, and their congruents in the group $\Gamma$.* Hence, with the expressions for $u$, $u'$ and $\mathfrak{R}$ defined as above, we may state

. THEOREM IV. *When $a$, $b$, $a'$, $b'$, $z'$ are fixed, so that no $u_\lambda'$ is a pole of $\mathfrak{R}(z)$, then*

$$\sum_{\lambda=0}^{\infty} [\mathfrak{R}\{(z_\lambda - a_\lambda)(z_\lambda - b_\lambda)\} - \mathfrak{R}\{(z_\lambda' - a_\lambda')(z_\lambda' - b_\lambda')\}]$$

*extended over the whole Fuchsian group represents an analytic function of $z$, which has the same poles as those of $\mathfrak{R}(u)$ and their congruents, and which has the unit circle as a natural boundary.*

It appears that in general these functions are not automorphic in the ordinary sense.

UNIVERSITY OF ILLINOIS.

* For a statement of formula (10) and its applications to trigonometric and elliptic functions see Schottky: "Ueber die Funktionenklasse, die der Gleichung $F\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) = F(x)$ genügt"; *Crelle*, vol. 143 (1913), pp. 1–24.

# PROFESSOR BÔCHER'S VIEWS CONCERNING THE GEOMETRY OF INVERSION.

BY PROFESSOR EDUARD STUDY.

In a recent paper (this Bulletin, volume 20, pages 185–200, January, 1914), Professor M. Bôcher sets forth what he thinks are sound principles for dealing with geometry, and more especially with the geometry of inversion. Herein I am glad to agree with him. The same and similar principles have been expounded, and applied, in a less elementary (but more comprehensive) manner by myself. Professor Bôcher's article may be looked upon as commenting on the import of the conception of *natural continua* introduced by me in 1903,* though he makes no use of this notion itself.

There seems to be, however, little agreement in other respects. Professor Bôcher quotes one of my articles ("Das Apollonische Problem," *Mathematische Annalen*, volume 49, 1897), merely stating that it is long and yet does not contain a word concerning the "region at infinity."† The first of these assertions is right as a matter of course, assuming a suitably chosen standard of length, and as to the second I will not quarrel with my critic. What I am concerned with is merely the inference which the author leaves to his readers. This inference would appear to be that I had been thoughtless, or careless, or regardless enough to publish my results in an embryonic state of development. It will inevitably be understood that all my theorems are "true in general" only, and consequently incorrect. I am found guilty of having committed an error to which, in recent years, I have myself objected often and strongly. I am caught in my own trap, and no mistake. There will be possibly some people who will enjoy this. But it cannot reasonably be expected that I should be one of their number. Therefore I beg to point out a few trifling circumstances that apparently have not been appreciated by my critic.

1. Let us supply the missing (but certainly indispensable)

* In my book, Geometrie der Dynamen, §§ 27, 28. This seems to have escaped Professor Bôcher's notice. He might have been aware of it, though, for he found a reference in connection with his own topic in a paper by H. Beck.

† Bôcher's own term is "the infinite region."

definitions, say, by the simple expedient of inserting a reference to Professor Bôcher's book of 1894. Then the haziness to which my critic objects will disappear at once and throughout. This remark, obvious as it is, does not seem to have occurred to Professor Bôcher; otherwise he ought to have mentioned it, for it alters the matter considerably. Probably he and I wrote for different sets of readers. I must, for some reason or other, have presupposed what Professor Bôcher explains. If my critic disapproves of this, it seems to me that he might have made clear the true trend of his objection, *which is irrelevant*, instead of making his readers believe that I had sinned against the first laws of logic, an intimation to which I cannot be indifferent. Professor Bôcher, who does not go far in the way of applications himself, might even have quoted my Apollonian paper as an illustration of the principles that underlie his doctrine; but, as a matter of fact, he meant to use it as a deterring example of their neglect, and only as such.

2. My fault (if fault it is) thus consisted in supposing notions widely known (and accepted by competent judges) which in reality perhaps were not so.* Why I held such an opinion is easily explained. The ideas in question are very simple. From the standpoint of Klein's Program of 1872 they appear almost as matters of course. To me they have been familiar since about 1884, the time when I was a young student under Professor Klein, who in later years was Professor Bôcher's teacher also, and to whom also Professor Bôcher, as he says himself, is indebted for the same ideas. It would seem to follow from this that the public I had in mind was likely to exist; and so it did, as is shown by the example of Professor Bôcher himself. But had it been nonexistent, I could still claim the right to write for it in the manner I did. What does not exist today may be common tomorrow. The first footnote in my paper shows that, notwithstanding its deplorable length, it is an extract only, and by no means intended for beginners.

3. I stated that I look upon the geometry of inversion as forming a *counterpart* (*Seitenstück*) *to projective geometry* (pages 498, 528), and I have treated it as such throughout. The idea apparently underlying Professor Bôcher's criticism,

* In my book of 1913 the point-continuum of the geometry of inversion is mentioned as a commonly accepted notion (page 282).

that in spite of this attitude (which was opposed to the habits of my predecessors, Darboux, for example) I might have operated in this projective domain seems quite absurd.

4. A piece of research in projective geometry is not necessarily defective if its author does not speak of points at infinity. The same holds good for the geometry of inversion.

5. That my point of view was *exactly* the one recommended by Professor Bôcher can be conclusively demonstrated. I say (page 539): "Um diese Aufgabe (das Apollonische Problem auf der Kugel) zu lösen, haben wir weder in Formeln noch Constructionen *auch nur die geringste Aenderung* anzubringen." Then there must be a one-to-one correspondence between the plane I operated in (the "plane of inversion") and the sphere, without any exceptional points.

Finally a remark may be made that has nothing to do with the subject at variance. The terms region and points *at infinity*, though very convenient, are apt to cause misunderstandings. It should be noticed that the (euclidean) distance between a point "at infinity" and an ordinary point is not necessarily infinite. Under certain conditions, the investigation of which I leave to the reader, it is indeterminate. The similar terms used in projective geometry require a similar comment.

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I do not care to follow Professor Study into the field of personal recrimination. Whether my article was calculated to be of use is neither for him nor for me to say. What it aimed to accomplish is clearly set forth in its first twelve lines, and this aim would not have been affected by the insertion, demanded by him, of a reference to his book. This reference would, however, as a matter of course have been made if I had been acquainted with the passage. The very brief allusion to the paper on the Apollonian Problem is the merest incident in the course of my article. In spite of Professor Study's remarks, the autobiographic part of which seems to me irrelevant, I see no substantial alteration which should be made in my words. I never said or implied, thought or wished the reader to think that this paper is too long. I do think it of sufficient length to warrant the expectation that the author should state explicitly and exactly what he was talking about. Various inferences, correct or incorrect, might be drawn from his failure to do so. I drew none.

MAXIME BÔCHER.

## THE DAVIS CALCULUS.

THE attack on our Calculus in the June BULLETIN perhaps calls for a word of protest. It is some consolation to know that the book has attractive elements, even though it is not stated what they are. Pleasant also is it that the reviewer recognizes that we did try to introduce interest into the subject. To get the student interested is certainly an important matter; and it is our belief that a text can and should help a teacher in doing so. The exciting of interest, however, is but a means to an end, that end being the stimulation of accurate thought on the many and varied applications of the calculus.

Mayhap the reviews of the book in the *American Mathematical Monthly*, in the *Bulletin* of the Society for Promoting Engineering Education, and in *Science* were far too favorable; yet, taken altogether, they would seem to indicate that we had been reasonably successful in carrying out our ideas. Still more satisfactory is it to know that of the seventy-odd institutions who will use the book for the coming year, a majority have already used it for three years.

Might we indeed possibly suggest that a review, to be valuable, should be accurate and judicial, not marred by exaggeration, by guesses as to how the book was written and what its reception will be, or by remarks on visionary ideals that have nothing to do with the use of the book?

ELLERY W. DAVIS.

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## NOTES.

THE July number (volume 16, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Sur les fonctionnelles bilinéaires," by M. FRÉCHET; "Oriented circles in space," by D. F. BARROW; "A new isosceles triangle solution of the three body problem," by D. BUCHANAN; "Surfaces $\Omega$ and their transformations," by L. P. EISENHART; "The general theory of congruences," by E. J. WILCZYNSKI; "On matrices whose coefficients are functions of a single variable," by J. H. M. WEDDERBURN; "Conformal classification of analytic arcs or elements: Poincaré's local problem of conformal geometry," by E. KASNER; "Extensions of Descartes' rule of signs connected with a problem suggested by Laguerre," by D. R. CURTISS; "On parastrophic algebras," by J. B. SHAW.

THE July number (volume 37, number 3) of the *American Journal of Mathematics* contains: "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface," by G. M. GREEN; "Linear combinants of systems of binary forms, with the syzygies of the second degree connecting them," by W. F. SHENTON; "Elimination d'une inconnue entre plusieurs équations algébriques," by M. STUYVAERT; "Congruences associated with a one-parameter family of curves," by R. D. BEETLE; "Plane sextic curves invariant under birational transformations," by A. HELEN TAPPAN.

THE academy of sciences of Paris has recently announced the following awards of prizes in the mathematical sciences:

Francoeur prize (fr. 1500) to Professor M. J. MARTY of the Lycée Alby, who has since been killed in battle.

Poncelet prize (fr. 2000) to Professor M. RABUT, emeritus professor in the Ecole des ponts et chaussées.

Boileau prize (fr. 1300) to Professor U. PUPPINI, of the University of Bologna.

AT the meeting of the Edinburgh mathematical society on June 11 the following papers were read: "On spheroidal harmonics and allied functions," by G. B. JEFFERY; "Determinants connected with the periodic solutions of Mathieu's equation," by A. G. BURGESS; "On the oscillation functions derived from a discontinuous function," by L. R. FORD; "The angle between two lines in trilinears," by W. L. MARR.

THE Helvetian society of natural scientists held its annual meeting at Geneva, September 12–15, this being its centenary celebration. The mathematical section was presided over by Professor H. FEHR.

A NEW edition of de Morgan's Budget of Paradoxes, edited by DAVID EUGENE SMITH with extended biographical, historical, and explanatory notes, has just been published by the Open Court Publishing Company.

The following advanced courses in mathematics are offered at the Italian universities during the academic year 1915–1916:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Classical problems of celestial mechanics, three hours.—By Professor L. DONATI: Modern electromagnetic theories. Thermodynamics and its relation to radiations; hypothesis of quanta, three hours.—By Professor F. ENRIQUES: Geometric theory of algebraic equations and functions, three hours.—By Professor S. PINCHERLE: Functional calculus. Integral equations and applications, three hours.

UNIVERSITY OF CATANIA.—By Professor E. DANIELE: Vibrations of elastic bodies. Theory of sound, four hours.—By Professor G. PENNACCHIETTI: Dynamics of rigid bodies (advanced part). Elasticity. Viscous fluids, four hours.—By Professor C. SEVERINI: Partial differential equations, four hours.—By ——: Higher geometry, three hours.

UNIVERSITY OF GENOA.—By Professor C. C. LEVI: Selected topics in the theory of partial differential equations, four and one half hours.—By Professor G. LORIA: Numerative geometry, three hours.—By Professor O. TEDONE: Electromagnetic theory of light, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics: the middle ages, three hours.—By Professor A. DEL RE: $n$-dimensional analysis of Grassmann, with application to mechanics of the spaces of constant curvature, four and one half hours.—By Professor R. MARCOLONGO: Fourier's series with several applications, three hours.—By Professor D. MONTESANO: Theory of birational correspondences of three-dimensional space, three hours.—By Professor E. PASCAL: Analytic functions and selected topics of mathematical analysis, three hours.—By Professor L. PINTO: Geometrical optics. Theory of optical instruments, three hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Functions of a complex variable; elliptic functions; integral equations, four hours.—By Professor A. COMESSATTI: Projective and descriptive geometry of hyperspaces, three hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Mechanics of continuous media: the technical, classical and relativistic point of view, four and

one half hours.—By Professor G. RICCI: Absolute differential calculus. General theory of elasticity, four hours.—By Professor F. SEVERI: Algebraic varieties from the point of view of reality, four hours.—By Professor A. SIGNORINI: Technical applications of elasticity, three hours.—By Professor A. TONOLO: Partial differential equations of the second order, three hours.—By Professor G. VERONESE: Geometrical applications of the theory of sets, four hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Calculus of variations, three hours.—By Professor M. DE FRANCHIS: Hyperelliptic surfaces and Picard's varieties, three hours.—By Professor M. GEBBIA: Mechanics of continuous bodies. Potential. Hydrodynamics. External acoustics, four and one half hours.—By ———: Mechanics (advanced part), three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Abelian integrals with geometrical applications mainly to correspondences between algebraic curves, three hours.—By Professor C. BOMPIANI: Geometry of numbers. Diophantine approximation, three hours.—By Professor U. CISOTTI: Electricity and magnetism, three hours.—By Professor F. GERBALDI: Elliptic functions, three hours.—By Professor G. VIVANTI: General theory of analytic functions, three hours.

UNIVERSITY OF PISA.—By Professor C. BERTINI: Geometry on a curve with algebraic and transcendental method, three hours.—By Professor L. BIANCHI: General theory of surfaces. Applicability. Rolling, four and one half hours.—By Professor U. DINI: Analytic representation of functions both in the real and in the complex field, four and one half hours.—By Professor G. A. MAGGI: Principles of analytic mechanics. Harmonic functions. Topics in hydrodynamics, four and one half hours.—By Professor P. PIZZETTI: Theory of the shape and of the motion of rotation of the planets, four and one half hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometrical applications of calculus, three hours.—By Professor G. CASTELNUOVO: Geometry of algebraic varieties, three hours.—By Professor U. CRUDELI: Introduction to the

advanced study of electricity, three hours.—By Professor V. VOLTERRA: Electricity and magnetism, three hours. Differential, integro-differential and derivative-functional equations of mechanics, three hours.—By ————: Theory of functions of a complex variable. Elliptic functions, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Equilibrium-figures of rotating fluid masses, three hours.—By Professor G. FUBINI: The modern advances of calculus. Application to expansions in series, to calculus of variations, to integral equations, three hours.—By Professor C. SEGRE: Topics in differential geometry, three hours.—By Professor C. SOMIGLIANA: Mechanical and electromagnetic optics, three hours.

PROFESSOR M. DE FRANCHIS, of the University of Catania, has accepted the professorship of higher geometry at the University of Palermo, as successor to the late Professor Guccia.

PROFESSOR E. BERTINI, of the University of Pisa, has been elected national member of the royal academy of Turin.

PROFESSORS F. ENRIQUES, of the University of Bologna, T. LEVI-CIVITA, and F. SEVERI, of the University of Padua, have been elected corresponding members of the royal institute of Lombardy.

THE Italian society of sciences (the forty) has awarded its gold medal for 1914, to Professor E. PASCAL, of the University of Naples for his work concerning integrating factors of differential and of integro-differential equations.

PROFESSOR G. B. MATHEWS, of Bangor College, Wales, has received the honorary degree of doctor of laws from Cambridge University.

DR. E. HECKE, of the University of Göttingen, has been appointed associate professor of mathematics at the University of Basel.

DR. K. KNOPP, of the University of Berlin, has been promoted to an associate professorship of mathematics.

PROFESSOR C. MOSER has been appointed professor of the theory of insurance at the University of Bern.

PROFESSOR F. SCHILLING, of the technical school at Dantzig, has received the title of Geheimer Regierungsrat.

PROFESSOR H. S. WHITE, of Vassar College, has received the degree of doctor of laws from Northwestern University.

PROFESSOR R. M. BARTON, of Lombard College, has been appointed dean and acting president.

DR. H. C. GOSSARD, of the University of Oklahoma, has been promoted to an assistant professorship of mathematics.

DR. W. L. MISER, of the University of Minnesota, has been appointed assistant professor of mathematics in the University of Arkansas.

DR. DANIEL BUCHANAN, of Queen's University, Kingston, Ontario, has been promoted to an associate professorship of mathematics and appointed director of the college observatory.

MR. J. L. RILEY has been appointed professor of mathematics in the Oklahoma state normal school at Tahlequah.

AT the State University of Iowa, Miss S. E. CRONIN has been promoted to an assistant professorship of mathematics.

PROFESSOR G. H. CRESSE, of Middlebury College, has been appointed assistant professor of mathematics in the University of Arizona.

DR. G. H. GRAVES, of Columbia University, has been appointed instructor in mathematics in Purdue University.

AT Princeton University Mr. L. S. HILL, of the University of Montana, has been appointed instructor in mathematics. Dr. H. GALAJIKIAN, has resigned his instructorship.

DR. H. B. PHILLIPS, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

MR. T. DANTZIG has been appointed instructor in mathematics at the University of Indiana.

MR. C. E. NORWOOD has been appointed assistant in mathematics at Dartmouth College.

J. C. WILSON, professor of logic at Oxford University since 1889, died August 12, 1915. He was the author of "On Traversing Geometrical Figures," published by the Clarendon Press in 1905.

JOHN HOWARD VAN AMRINGE, emeritus professor of mathematics in Columbia University, died September 10, 1915, at the age of 80 years. He was an active member of the Columbia faculty for fifty years preceding his retirement in 1910. In 1894 he became dean of the college. He was a founder and the first president of the New York Mathematical Society, since reorganized as the American Mathematical Society.

JOHN K. SINCLAIR, emeritus professor of mathematics in the Worcester polytechnic institute, died September 12, 1915.

PROFESSOR C. A. VON DRACH, of the University of Marburg, died recently at Cassel in his seventy-sixth year.

PROFESSOR O. SIMONY, of the agricultural institute at Vienna, died April 6, 1915.

PROFESSOR N. V. SONIN, minister of education and member of the Petrograd academy of sciences, died February 27, at the age of 66 years.

PROFESSOR A. WERNICKE, of the technical school at Braunschweig, died March 30, at the age of 58 years.

# NEW PUBLICATIONS.

## I. HIGHER MATHEMATICS.

BAGNERA (G.). Lezioni di calcolo infinitesimale. Disp. 1–35. Palermo, tip. Matematica, 1914. 8vo. Pp. 1–280.

BALL (W. W. R.). Mathematical recreations and essays. 6th edition. New York, Macmillan, 1914. 8vo. 16+506 pp. $3.25

BALZINI (A.). Il problema della quadratura dêl circolo risoluto. Pisa, A. Macchi, 1915. 4to. 7pp.+3 tavole.

BROGGI (H.). Ecuaciones integrales lineales. La Plata, 1914. 36 pp.

BÜTZBERGER (F.). Ueber bizentrische Polygone, Steinersche Kreis- und Kugelreihen und die Erfindung der Inversion. 2ter Teil. Zurich, 1914. 8vo. 27pp.

CARL (A.). Zur Theorie der ebenen ähnlich-veränderlichen Systeme. (Diss.) Dresden, 1914.

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 2, Heft 5: G. Loria, Spezielle ebene algebraische Kurven von höherer als der vierten Ordnung. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 571–634. M. 2.40

ERLER (H.). Metrische Relationen an den vollständigen Figuren und am Kegelschnitt. Breslau, 1914.

FUSS (H.). Modulsysteme und höhere komplexe kommutative Zahlsysteme. Kiel, 1913. 8vo. 70 pp.

GRECO (D.). La quadratura del cerchio. Maglie, tip. Messapica di B. Canitano, 1915. 8vo. 15 pp.

HARMS (F.). Die Transfiguration ebener Kurven von rechtwinkligen cartesischen auf A-normalen- und A-Strahlbüschelkoordinaten. Rostock, 1913. 8vo. 45 pp.+3 Tafeln.

HINTZER (F.). Die durch einen Kreispunkt gehenden geodätischen Linien auf dem elliptischen Paraboloid. Friedeberg, 1914. 4to. 23 pp.

JENTZSCH (R.). Untersuchungen zur Theorie der Folgen analytischer Funktionen. (Diss.) Berlin, 1914.

KISSEL (G.). Ueber den von den Trisektionslinien eines Dreiecks umhüllten Kegelschnitt. Giessen, 1913. 8vo. 50 pp.

LANGHANS (C.). Einführung in die Differentialrechnung. Plön, 1914. 8vo. 34 pp. M. 1.50

LORIA (G.). See ENCYKLOPÄDIE.

MACMAHON (P. A.). Combinatory analysis. Vol. I. Cambridge, University Press, 1915. 8vo. 19+300 pp. 15s.

MEYER (C.). Ueber den ersten Differentialquotienten des logarithmischen Kurvenpotentials. Hamm, 1913. 8vo. 37 pp.

POTTHOF (R.). Eine von Lagebeziehungen unabhängige geometrische Beweismethode und deren Anwendung. Wanne, 1914. 4to. 13 pp.

RUDOLPHI (W.). Analytische Geometrie des Raumes in Verbindung mit darstellender Geometrie. 3ter Teil: Kreiskegel und Kreiszylinder. Neumünster, 1914. 4to. 29 pp.+11 Tafeln.

STARKE (P.). Die Apollonischen Sehnenvierecke des Dreiecks. Leipzig, 1914. 4to. 28 pp. M. 1.60

WINKLER (R.). Ueber die Bewegung affin-veränderlicher ebener Systeme. (Diss., Techn. Hochschule Dresden.) Borna-Leipzig, R. Noske, 1914.

## II. ELEMENTARY MATHEMATICS.

BAKER (A. E.). Commercial arithmetic. Chicago, Metropolitan Text Book Co., 1912. $1.00

BOHNERT (F.). Grundzüge der ebenen Geometrie. Leipzig, Göschen, 1915. 8vo. M. 2.80

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